CHAPTER SIX

6-2

The initial static equilibrium is characterized by a balance between the spring force and the object's weight: $mg = k\ell$. We denote the instantaneous position by $y(t)$, with $y = 0$ corresponding to the equilibrium position and with $y > 0$ corresponding to additional stretching. Once the object has been released, Newton's second law is given by

$$m \frac{d^2y}{dt^2} = mg - k(\ell + y) = -ky$$

That is, $a = -ky/m$, as desired. Therefore the body executes harmonic motion about its static equilibrium position; the period T of the motion is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{\ell}{g}}$$

6-4

(a) As shown in the solution for Exercise 6-2, the period of oscillation is given by $T = 2\pi\sqrt{\ell/g}$.

(b) Referring to Equation (6-28b), we see that the period of a pendulum of length ℓ is given by $2\pi\sqrt{\ell/g}$, which is equal to the oscillation period found in part (a).

6-7

(a) Using the general expression $x(t) = A\cos(2\pi\nu t + \delta)$, we find $v(t) = dx(t)/dt = -2\pi\nu A\sin(2\pi\nu t + \delta)$. Therefore $x(0) = A\cos\delta$ and $v(0\_ = -2\pi\nu A\sin\delta$. Solving for A and δ, we find

$$A = \frac{\sqrt{[2\pi\nu x(0)]^2 + [v(0)]^2}}{2\pi\nu}$$ and

$$\delta = \tan^{-1}\left[\frac{-v(0)}{2\pi\nu x(0)}\right]$$

Substituting the given values into these equations, we have

$$A = \frac{\sqrt{[2\pi\cdot 5\cdot 10]^2 + [-2\pi\cdot 50]^2}}{2\pi\cdot 5} = \sqrt{200} = 10\sqrt{2} = 14.1 \text{ cm}$$

and $\delta = \tan^{-1}\left[\frac{-(-2\pi \cdot 50)}{2\pi \cdot 5 \cdot 10}\right] = \frac{\pi}{4}$ rad

since the initial conditions imply that both $\sin\delta$ and $\cos\delta$ are positive. Therefore the displacement is given by $x(t) = (14.1 \text{ cm}) \cdot \cos(10\pi t + \pi/4)$. The velocity is given by

$x(t) = dx/dt = -(100\,\pi\sqrt{2}\text{ cm/s}) \cdot \sin(10\pi t + \pi/4)$. The acceleration $a(t) = -(1000\,\pi^2\sqrt{2}\text{ cm/s}^2) \cdot \cos(10\pi t + \pi/4)$.

(b) As found in part (a), the amplitude $A = 10\sqrt{2} = 14.1$ cm, and the phase angle $\delta = \pi/4$ rad.

(c) The maximum values are: $x_{max} = 10\sqrt{2} = 14.1$ cm, $v_{max} = 100\pi\sqrt{2} = 444$ cm/s, and $a_{max} = 1000\pi\sqrt{2} = 1.40 \times 10^4$ cm/s^2.

6-8

(a) Using $x(t) = A\cos(2\pi\nu t + \delta)$, we have $v(t) = -2\pi\nu A\sin(2\pi\nu t + \delta)$. Since $x(0) = 0$, we have $\delta = \pi/2$ or $3\pi/2$. Because $v(0) > 0$, we must have $\delta = 3\pi/2$. Then, with the given values $A = 2.00$ cm and $\nu = 3.00$ Hz, we obtain $x(t) = (2.00 \text{ cm})\cos(6.00\pi t + 3\pi/2)$,

$v(t) = (-12.0\pi \text{ cm/s})\sin(6.00\pi t + 3\pi/2)$, and

$a(t) = (-72.0\pi^2 \text{ cm/s}^2)\cos(6.00\pi t + 3\pi/2)$.

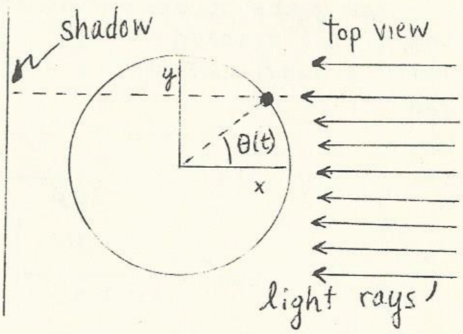
(b) At $t = 0.05$ s, $6.00\pi t + 3\pi/2 = 1.80\pi = 5.65$ rad.

Therefore, we find $x(0.05 \text{ s}) = (2.00 \text{ cm})\cos(1.80\pi) = 1.62$ cm,

$v(0.05 \text{ s}) = (-12.0\pi \text{ cm/s})\sin(1.80\pi) = 22.2$ cm/s,

and $a(0.05 \text{ s}) = (-72.0\pi^2 \text{ cm/s}^2)\cos(1.80\pi) = -575$ cm/s^2.

6-9



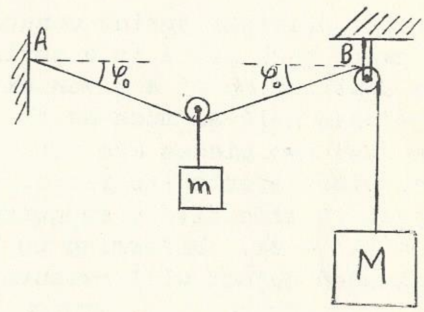
The shadow moves back and forth as the bear circles. We use a Cartesian coordinate system with the origin at the center of the circle, and with $\hat{x}$ pointing toward the spotlight. Taking $\hat{z}$ vertically upward, the instantaneous y coordinate of the bear gives y(t) for the shadow, as shown in the figure at right. We let $\theta(t)$ denote the bear's angular position (in radians), measured counter-clockwise from the positive x-axis. Then $d\theta(t)/dt = \pm v/r$, with the sign depending upon which way the bear is pedaling. (The positive sign corresponds to the direction shown in Fig. 6E-9.)

Letting $\theta(0) = \theta_o$, the angular position $\theta(t) = (\pm vt/r) + \theta_o$. But $y = r \sin\theta(t) = r \sin[(\pm vt/r) + \theta_o)] = r \cos[(\pm vt/r) + (\theta_o - \pi/2)]$. For counterclockwise pedaling, we see immediately that $\omega = v/r$. For clockwise motion (lower sign), we note that

$\cos[(-vt/r) + (\theta_o - \pi/2)] = \cos[vt/r - (\theta_o - \pi/2)]$, so again we have $\omega = v/r$. In either case, we have harmonic motion of amplitude r and angular frequency $\omega = v/r$.

6-10

(a) Yes, there is a unique equilibrium position. The body of mass M is in equilibrium when the tension T in the vertical string segment balances the weight: T = Mg. Since the pulleys are frictionless and the string is massless, all parts of the entire string have the same tension. Then a static equilibrium is possible when the object of mass m is midway between A and B as shown in the figure. The string segments A-m and m-B make an angle φ_o with the horizontal, such that $2T \sin \varphi_o = mg$. That is, the static equilibrium configuration has $\varphi_o = \sin^{-1}(m/2M)$. For the value M = 10 m given, $\varphi_o = \sin^{-1}(0.05) = 2.87^\circ$.



(b) Yes, the equilibrium is a stable one. Suppose that the object of mass M were pulled down slightly and then released from rest. If the object remained at rest or resumed its downward motion, that would imply a string tension T' equal to or less than Mg. Now because the string is massless and the pulleys are frictionless, all segments of the string must have the same tension T', even when the configuration is not one of static equilibrium. Then the net upward force on the other object (mass m) would be less than its weight: $2T' \sin\varphi' < 2(Mg)\cdot\sin\varphi' < 2T \sin \varphi_o = mg$. (Here we have used the fact that if the object of mass M descends, the angle φ must decrease.) Therefore the object of mass m must descend, but it cannot if the object of mass M is descending. We have arrived at a contradiction. Therefore the assumption that the object of mass M will remain or will resume its descent if it is pulled down and then released from rest must be incorrect. If the object of mass M is pulled down and then released, it will rise. A completely analogous argument shows that if the object of mass M is raised and then released, it will descend. That is, if the system is displaced (in either direction) from its static equilibrium configuration and is then released, it will move back toward its equilibrium configuration. This shows that the equilibrium is stable.

6-11

Let M be the object's mass; the mass of the board will be neglected. Also using a coordinate y (increasing downward) with y = 0 at equilibrium, Newton's second law gives

$$M \frac{d^2y}{dt^2} = Mg - C(d + y)$$

Here C represents the proportionality constant between the sag and the restoring force. Since Mg = Cd describes the static equilibrium, we find $M(d^2y/dt^2) = -Cy = -(Mg/d)y$, or $d^2y/dt^2 = -(g/d)y$. This is the equation of harmonic motion of angular frequency $\omega = \sqrt{g/d}$ and period $T = 2\pi/\omega = \underline{2\pi\sqrt{d/g}}$.

6-12

Let the original spring constant be k. When the spring is cut in half, each piece is a spring whose constant k' = 2k. (Under the application of a given external force, each half of the spring stretches half as much as the entire spring would have stretched.) When the two pieces are used side-by-side to support an object, each piece exerts its restoring force on that object. Thus, the object is effectively supported by a spring of spring constant k" = 2k' = 4k. Referring to Exercise 6-2, we know that the suspended object will execute harmonic motion of period

$T'' = 2\pi\sqrt{m/k''} = 2\pi\sqrt{m/4k} = (2\pi\sqrt{m/k})/2 = T/2.$

6-13

Since the springs are assumed to be massless, under the application of a given stretching force F, the compound spring stretches by distance $x = x_A + x_B$, where $x_A = F/k_A$ and $x_B = F/k_B$. Therefore

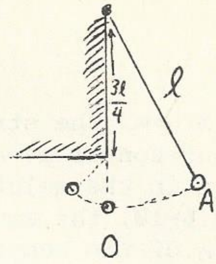
$$x = \frac{F}{k_A} + \frac{F}{k_B} = F\left(\frac{k_A + k_B}{k_A k_B}\right) \quad \frac{F}{k_{eff}}$$

The effective spring constant $k_{eff} = k_A k_B/(k_A + k_B)$. The oscillation period $T = 2\pi\sqrt{m/k_{eff}}$. With the given values, we find k_{eff} = (2000)(1000)/(3000) = (2000/3) N/m. The period of oscillation of the 50-kg mass is therefore $2\pi\sqrt{(50\cdot 3)/2000} = 2\pi\sqrt{0.075}$ = <u>1.72 s</u>.

6-14

The situation is shown in the figure at the top of the next page. When the bob is located to the left of point O, the obstacle acts as the pivot point, and the pendulum is effectively of length

$\ell' = \ell/4$. When the bob is located to the right of point O, the pendulum has its full length ℓ. If the bob is released from point A, it will return after a total of one-half cycle of swinging as a pendulum of length ℓ, and one-half cycle as a pendulum of length $\ell/4$. The elapsed time is therefore $(1/2)(2\pi\sqrt{\ell/g}) + (1/2)(2\pi\sqrt{\ell/4g})$. Since $2\pi\sqrt{\ell/g} = T$, the elapsed time is $T/2 + T/4 = \underline{3T/4}$.



6-15

(a) The statement given may be summarized by saying that if N bacteria are present at time t, then the increase ΔN in the number during a small succeeding time interval Δt is given by $\Delta N = RN\Delta t$, where R is a positive constant. Therefore the total number N(t) at any given time t obeys the equation

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} \equiv \frac{\Delta N}{\Delta t} = RN$$

Taking the limit as $\Delta t \to 0$, we find $dN/dt = RN$, which is the desired equation.

(b) Since the derivative of an exponential function is proportional to the function itself, we try $N(t) = Ae^{Bt}$. When this is substituted into the differential equation, we find $BAe^{Bt} = RAe^{Bt}$, which is valid for B = R and for any value of A. If we let N_o represent the number of bacteria present at t = 0, we have $A = N_o$. Then $\underline{N(t) = N_o e^{Rt}}$.

6-16

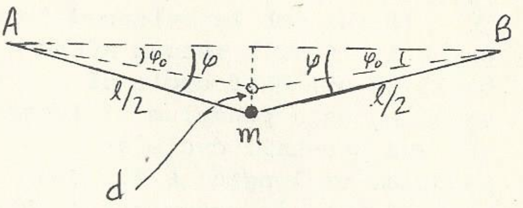
(a) The statement may be summarized by saying that if N(t) nuclei are present at time t, then the number $N(t + \Delta t)$ present at time $t + \Delta t$ is given by $N(t + \Delta t) - N(t) = -RN(t)\Delta t$, where R is a positive constant. This leads to the equation

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = -RN(t)$$

The limit of this equation is $dN/dt = -RN$, as desired.

(b) As in the previous exercise, we try $N(t) = Ae^{Bt}$. We find that we must use B = -R. Then $\underline{N(t) = N_o e^{-Rt}}$, where N_o is the number of undecayed nuclei at t = 0.

6-17



Because $M \gg m$, the string is nearly horizontal at equilibrium. [As shown in the solution for Exercise 6-10, the equilibrium angle φ_o of the supporting strings is $\sin^{-1}(m/2M)$.] Referring to the figure at the right, if the object of mass m is pushed down an additional distance d, the strings exert a combined upward force that exceeds the object's weight by approximately $2T(\sin\varphi - \sin\varphi_o)$. (A more detailed analysis shows that the change in tension can be neglected.) Because both φ_o and φ are small angles, $\sin\varphi - \sin\varphi_o \simeq \varphi - \varphi_o \simeq \sin(\varphi - \varphi_o) \simeq d/(\ell/2)$. Therefore Newton's second law implies that

$$ma \simeq -2T(\sin\varphi - \sin\varphi_o)$$

$$\simeq -2T[d/(\ell/2)] \simeq -(4T/\ell)d$$

Since T = Mg, the displayed equation implies harmonic motion of angular frequency $\omega \simeq \sqrt{4Mg/m\ell} \simeq 2\sqrt{Mg/m\ell}$ and period $T = 2\pi/\omega \simeq \pi\sqrt{m\ell/Mg}$.

6-18

(a) As long as the string remains taut, the body is subjected to a restoring force which varies linearly with vertical displacement from the position of static equilibrium. In this case the only effect of the string is to communicate the spring force to the body. The motion of the body is harmonic motion of amplitude A and frequency $\omega = \sqrt{k/m}$. The maximum magnitude of the acceleration (in either direction) is $\omega^2 A = kA/m$.

(b) If the tension in the string is T(t), the downward acceleration $a(t) = g - [T(t)/m]$. Therefore the downward acceleration cannot exceed g if the string is to remain taut ($T > 0$). The corresponding maximum amplitude $A_{max} = mg/k$. Notice that this amplitude is equal to the amount of stretching in the spring at equilibrium, which is consistent with our intuition.

(c) With m = 0.10 kg and k = 10 N/m, we find $A_{max} = (0.10)(9.80)/10 = 9.80 \times 10^{-2}$m = 9.80 cm.

6-19

(a) If the amplitude of oscillation of the massive block is A, then its maximum acceleration is $\omega^2 A = 4\pi^2\nu^2 A$. In order for the smaller block to follow this motion, the frictional force must be able to produce the above acceleration. The maximum acceleration that static friction can provide in this case is $\mu_s g$. Therefore we need $4\pi^2 \nu^2 A \leq \mu_s g$, so the maximum amplitude A_{max} $= \mu_s g/4\pi^2\nu^2$.

(b) With $\nu = 3.0$ Hz and $\mu_s = 0.60$, we find A_{max} $= (0.60)(9.80)/4\pi^2(3.0)^2 = 1.65 \times 10^{-2}$ m = 1.65 cm.

6-20

(NOTE: In parts (b) - (d) below, more than three significant figures are retained. This is done so that three-figure accuracy can be obtained in part (e). This procedure also requires us to assume that the accuracy of the given information extends beyond three places.)

(a) The appropriate expression in this case is $x(t)$ $= A\cos(\omega_0 t + \delta)$, where $\omega_0 \equiv \sqrt{\alpha} = 1.00\ s^{-1}$. Then $v(t)$ $= -\omega_0 A\sin(\omega_0 t + \delta)$. The initial conditions $x(0) = 0$ and $v_0 = 5.00$ imply that $A = 5.00$ m and $\delta = -\pi/2$ rad. Therefore $x(t) = 5.00\cos(\omega_0 t - \pi/2)$ = 5.00(sin t) m.

(b) In the case of light damping, the expression is

$$x(t) = Ae^{-\beta t/2}\cos\left(\sqrt{\omega_0^2 - \frac{\beta^2}{4}}\; t + \delta\right)$$

Letting $\omega \equiv \sqrt{\omega_0^2 - \frac{\beta^2}{4}}$,

$$\text{Then } v(t) = \frac{-\beta}{2} Ae^{-\beta t/2}\cos(\omega t + \delta)$$

$$-Ae^{-\beta t/2}\omega\sin(\omega t + \delta).$$

The initial conditions require that $\delta = -\pi/2$ and $\omega A = 5.00$ m. For $\beta = 0.10\ s^{-1}$, we find $\omega = 0.9987\ s^{-1}$ and $A = 5.006$ m. Therefore $x(t) = 5.006\, e^{-0.05t}\sin(0.9987t)$ m. For $\beta = 1.0\ s^{-1}$, we find $\omega = 0.8660\ s^{-1}$ and $A = 5.774$ m. Then $x(t)$ $= 5.774\, e^{-0.5t}\sin(0.8660t)$ m.

(c) In the case of critical damping, the displacement is given by $x(t) = (A + Bt)\, e^{-\beta t/2}$, so that the velocity is given by

$$v(t) = \frac{-\beta}{2}(A + Bt)\, e^{-\beta t/2} + B\, e^{-\beta t/2}$$

The initial conditions imply that $A = 0$ and $B = 5.00$ m. Therefore $x(t) = 5.00te^{-t}$ m.

(d) In the case of heavy damping, the expression for the displacement is

$$x(t) = (Ae^{\Gamma t} + Be^{-\Gamma t})e^{-\beta t/2}$$

where we have set $\Gamma \equiv \sqrt{\beta^2/4 - \omega_o^2}$. The velocity is given by

$$v(t) = \Gamma\left(Ae^{\Gamma t} - Be^{-\Gamma t}\right)e^{-\beta t/2} - \frac{\beta}{2}\left(Ae^{\Gamma t} + Be^{-\Gamma t}\right)e^{-\beta t/2}$$

The initial conditions imply that B = -A and A = $5/2\Gamma$ m. For $\beta = 3.00\ s^{-1}$, we find $\Gamma = \sqrt{5/4} = \sqrt{5}/2$. Therefore A = $\sqrt{5}$ = 2.236 m, $(\Gamma - \beta/2) = -0.38197\ s^{-1}$ and $(-\Gamma - \beta/2) = -2.61803\ s^{-1}$,

so that

$$x(t) = 2.236\left(e^{-0.38197t} - e^{-2.61803t}\right)\ m$$

For $\beta = 10.0\ s^{-1}$, we find $\Gamma = \sqrt{24} = 2\sqrt{6}$. Then A = $5/(4\sqrt{6})$ = 0.5103 m, $(\Gamma - \beta/2) = -0.10102\ s^{-1}$, and $(-\Gamma - \beta/2) = -9.89898\ s^{-1}$, so that

$$x(t) = 0.5103\left(e^{-0.10102t} - e^{-9.89898t}\right)\ m$$

(e) Undamped motion. In this case $x_M = \max[x(t)] = v_o/\omega$, so that $(x_M\sqrt{\alpha}/v_o)$ = 1.00.

Lightly damped motion. The equation for v(t) displayed in part (b) shows that x(t) attains its maximum value when $\tan(\omega t) = 2\omega/\beta$ or when $\omega t = \tan^{-1}(2\omega/\beta)$. For $\beta = 0.10\ s^{-1}$, $\omega t_M = 1.520775$ and

$$\frac{x_M\sqrt{\alpha}}{v_o} = \left(\frac{A\sqrt{\alpha}}{v_o}\right)e^{-\frac{\beta}{2\omega}(\omega t_M)}\sin(\omega t_M)$$

$$= \left(\frac{5.006}{5}\right)\left(e^{\frac{-1.520775}{19.97498}}\right)(0.9987)$$

$$= \underline{0.927}$$

For $\beta = 1.00\ s^{-1}$, $\omega t_M = 60^o = 1.047197$ rad, and

$$\frac{x_M\sqrt{\alpha}}{v_o} = \left(\frac{5.774}{5}\right)\left(e^{\frac{-1.047197}{1.732051}}\right)(0.8660)$$

$$= \underline{0.546}$$

Critically damped motion. The equation for v(t) given in part (b), with A = 0, shows that x(t) is a maximum when $1 - \beta t/2 = 0$ or when $t_M = 2/\beta = 1.00$ s.

$$\frac{x_M\sqrt{\alpha}}{v_o} = \left(\frac{5.00}{5}\right) t_M e^{-t_M} = (1)(1)e^{-1} = \underline{0.368}$$

Heavily damped motion. Close examination of the equations given in part (c), with $A = -B = 5/2\Gamma$, shows that

$$x(t) = (5/\Gamma)\sinh(\Gamma t)e^{-\beta t/2}$$

where $\sinh(\Gamma t) \equiv (e^{\Gamma t} - e^{-\Gamma t})/2$. (See note on the hyperbolic functions at end of this solution.) Then the velocity is given by

$$v(t) = \left(\frac{5}{\Gamma}\right)\left[\Gamma\cosh(\Gamma t) - \frac{\beta}{2}\sinh(\Gamma t)\right] e^{-\beta t/2}$$

Here $\cosh(\Gamma t) \equiv (e^{\Gamma t} + e^{-\Gamma t})/2$. Therefore x(t) attains its maximum when $[\sinh(\Gamma t)/\cosh(\Gamma t)] \equiv \tanh(\Gamma t) = 2\Gamma/\beta$. That is, $\Gamma t_M = \tanh^{-1}(2\Gamma/\beta)$. Then the desired ratio is given by

$$\frac{x_M\sqrt{\alpha}}{v_o} = \left(\frac{1}{\Gamma}\right)\sinh(\Gamma t_M)e^{-\frac{\beta}{2\Gamma}(\Gamma t_M)}$$

For $\beta = 3.00\ s^{-1}$, we have $\Gamma = \sqrt{5}/2$, and we find $(2\Gamma/\beta) = 0.745356$ and $\Gamma t_M = 0.962424$. Therefore

$$\frac{x_M\sqrt{\alpha}}{v_o} = \left(\frac{2}{\sqrt{5}}\right)\sinh(0.962424)e^{\frac{-0.962424}{0.745356}}$$

$$= \underline{0.275}$$

For $\beta = 10.0\ s^{-1}$, we have $\Gamma = 2\sqrt{6}$, and we find $(2\Gamma/\beta) = 0.979796$ and $\Gamma t_m = 2.292432$. Therefore

$$\frac{x_M\sqrt{\alpha}}{v_o} = \frac{1}{2\sqrt{6}}\sinh(2.292432)e^{\frac{-2.292432}{0.979796}}$$

$$= \underline{0.096}$$

To summarize, for damping constants β = 0, 0.10, 1.00, 2.00, 3.00, and 10.0 s^{-1}, we find ratios $(x_M\sqrt{\alpha}/v_o)$ = 1.00, 0.927, 0.546, 0.368, 0.275, and 0.096, respectively.

(NOTE: The hyperbolic functions and some of their properties are utilized in the solution presented for part (e). It is of course possible to avoid using the hyperbolic functions, but the solution is even more tedious than the one provided here.)

6-21

(a) For convenience, we let the spring lie along the x-axis, so that the entire motion is along x. We also let $t = 0$ when the bodies are released, and we take the initial position of body #1 as the origin: $x_1(0) = 0$. Then $x_2(0) = \ell_o - A$, where ℓ_o is the relaxed length of the spring. [We could have taken $x_2(0) = -(\ell_o - A)$; it does not affect the physics of this exercise.] Newton's second law, as applied to body #1, reads

$$m_1 \frac{d^2 x_1}{dt^2} = k[(x_2 - x_1) - \ell_o]$$

For body #2, Newton's second law reads

$$m_2 \frac{d^2 x_2}{dt^2} = -k[(x_2 - x_1) - \ell_o]$$

Together these two equations imply that

$$\frac{d^2}{dt^2}(x_2 - x_1 - \ell_o) = -\frac{k}{m_2}(x_2 - x_1 - \ell_o) - \frac{k}{m_1}(x_2 - x_1 - \ell_o)$$

$$= -\frac{k}{\mu}(x_2 - x_1 - \ell_o)$$

where $\mu \equiv m_1 m_2/(m_1 + m_2)$. The last displayed equation shows that the spring extension $(x_2 - x_1 - \ell_o)$ varies sinusoidally with a period $T = 2\pi\sqrt{\mu/k}$. The law of conservation of momentum implies that $m_1 v_1 + m_2 v_2 = 0$, so the velocities are related by $v_2/v_1 = -m_1/m_2$. The amplitudes A_1 and A_2 of the motions of bodies 1 and 2 are related by $A_2/A_1 = m_1/m_2$. More specifically, the initial conditions $[x_2(0) - x_1(0)] - \ell_o = -A$ and $[v_2(0) - v_1(0)] = 0$ imply that

$$x_2(t) - x_1(t) - \ell_o = -A\cos(\omega t) \quad \text{and}$$

$$v_2(t) - v_1(t) = \omega A \sin(\omega t)$$

where $\omega \equiv \sqrt{k/\mu}$. Letting $m_1 + m_2 = M$, we find

$$v_1(t) = -(m_2/M)\omega A \sin(\omega t) \quad \text{and}$$

$$v_2(t) = (m_1/M)\omega A \sin(\omega t)$$

When these velocity equations are integrated, subject to the initial conditions, we find

$$x_1(t) = -(m_2/M)A(1 - \cos\omega t)$$

$$x_2(t) = (\ell_o - A) + (m_1/M)A(1 - \cos\omega t)$$

(b) For $m_1 \gg m_2$, the $\mu \simeq m_2$, so the period approaches $2\pi\sqrt{m_2/k}$. The amplitude ratio A_2/A_1 increases without limit, which means that body #2 is executing essentially all of the motion: the amplitude $A_2 = m_1A/M \rightarrow A$. For $m_1 \ll m_2$, the same limits are approached, with the subscripts 1 and 2 interchanged everywhere.

(c) For $m_1 = m_2 = m$, we find $\mu = m/2$, and the period $T = 2\pi\sqrt{\mu/k} = 2\pi\sqrt{m/2k} = (1/\sqrt{2})(2\pi\sqrt{m/k})$. The velocities are equal and opposite: $v_2(t) = -v_1(t)$, with $v_1(t)$ given by

$$v_1(t) = -(\omega A/2)\sin(\omega t)$$

where $\omega = \sqrt{2}(\sqrt{k/m})$. The positions are

$$x_1(t) = -(A/2)(1 - \cos\omega t) \quad \text{and}$$

$$x_2(t) = (\ell_o - A) + (A/2)(1 - \cos\omega t)$$

The oscillation period $(1/\sqrt{2})(2\pi\sqrt{m/k})$ is shorter than that of a single object of mass m oscillating an anchored spring of constant k. This is so because when the spring pushes on the two-body system, both bodies are accelerated, so the compressions are removed more rapidly. (The extensions are also removed more rapidly.)

6-22

(a) Let x_1 and x_2 be the coordinates of objects 1 and 2, measured from the initial equilibrium (with the spring relaxed) and with $x_1 > 0$ corresponding to a stretched spring. If the two blocks move as a unit, then $x_1 = x_2 = x$, and Newton's second law reads $(M_1 + M_2)d^2x/dt^2 = -kx$. The motion is a harmonic oscillation of period $T = 2\pi\sqrt{(M_1 + M_2)/k} = \underline{2\pi\sqrt{(1 + \alpha)M_1/k}}$.

(b) The required maximum magnitude of the acceleration d^2x_2/dt^2 of block 2, in order for it to move with block 1, is given by ω^2A. Since friction is the only horizontal force acting on block 2, the two blocks can move as a unit only if $M_2\omega^2A \leq \mu_sM_2g$. This implies a maximum amplitude $A_{max} = \mu_s(M_1 + M_2)g/k = \underline{(1 + \alpha)M_1g\mu_s/k}$.

(c) With $k = 6.0$ N/m, $M_1 = 1.0$ kg, $\alpha = 0.50$, and $\mu_s = 0.40$, we find $T = 2\pi\sqrt{(1.5)(1.0)/(6.0)} = \pi = \underline{3.14 \text{ s}}$. The maximum amplitude A_{max} = (1.5)(1.0)(9.8)(0.40)/(6.0) = 0.98 m.

6-23

(a) We let t = 0 denote the instant of firing, and t_b, t_m, and t_f represent the instants of burnout, maximum altitude, and impact, respectively. The burnout altitude $y_b = \frac{1}{2} a_y t_b^2$ and the speed at burnout $v_b = a_y t_b$. Then $t_m - t_b = v_b/g$ and the peak altitude $y_m = y_b + \frac{1}{2} v_b (t_m - t_b)$. Finally, we have $t_f = t_m + \sqrt{2y_m/g}$. With $a_y = 3.0g = 29.4\ m/s^2$ and $t_b = 15$ s, we find $y_b = 3.31 \times 10^3$ m and $v_b = 4.41 \times 10^2$ m/s. Then $t_m - t_b = 45$ s, so that $t_m = 60$ s. The maximum altitude is given by

$$y_m = 3.31 \times 10^3 \text{ m} + (0.5)(4.41 \times 10^2 \text{ m/s})(45 \text{ s})$$
$$= 1.32 \times 10^4 \text{ m}$$

Finally we obtain

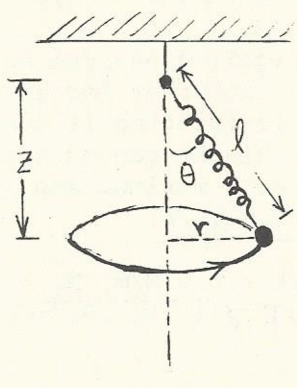
$$t_f = t_m + \sqrt{\frac{2y_m}{g}}$$
$$= 60 \text{ s} + \sqrt{\frac{2(1.32 \times 10^4 \text{ m})}{(9.80 \text{ m/s}^2)}}$$
$$= \underline{112 \text{ s}}$$

This consists of 15 s of powered rise, 45 s of upward coasting, and 52 s of descent.

(b) The effective gravity g_{eff} within the rocket is equal to the rocket's acceleration a_y (positive upward), plus g, the magnitude of earth's gravity. During the burn, $a_y = 3g$, so that $g_{eff} = 4.0$ g. After burnout, $a_y = -g$, so that $g_{eff} = 0$. The period of the rocket's pendulum clock is proportional to $(g_{eff})^{-1/2}$, so it runs at twice the correct rate during the burn and then stops running entirely. Therefore it records a "duration" of 2 x 15 s = 30 s for the flight.

6-24

We use the notations indicated in the figure at the right. The motion shown is counterclockwise as seen from above. (Evidently for each such motion a corresponding clockwise motion is also possible.) In order that the motion be in a horizontal plane, the net force along z must vanish, so we obtain



$$0 = mg - k(\ell - \ell_o)\cos\theta$$

The only possible horizontal circular motion has constant speed v; Newton's second law gives the required centripetal force:

$$-m\omega^2 r = -m\omega^2(\ell \sin\theta) = -k(\ell - \ell_o)\sin\theta$$

where $\omega \equiv v/r$ is the angular frequency. We now use the displayed equations to express ω, θ, r, and z in terms of ℓ, the length of the spring. The second displayed equation shows that

$$\omega = \sqrt{\frac{k}{m}}\sqrt{1 - \left(\frac{\ell_o}{\ell}\right)}$$

while the first displayed equation shows that

$$\cos\theta = \frac{mg}{k(\ell - \ell_o)} \qquad \text{or}$$

$$\theta = \cos^{-1}\left[\frac{mg}{k(\ell - \ell_o)}\right]$$

Then the radius r of the circular motion is given by

$$r \equiv \ell \sin\theta = \ell\sqrt{1 - \left[\frac{mg}{k(\ell - \ell_o)}\right]^2}$$

and the vertical coordinate z is given by

$$z \equiv \ell \cos\theta = \frac{mg}{k(1 - \ell_o/\ell)} = \frac{g}{\omega^2}$$

It is instructive to examine limiting cases. The range of possible spring lengths ℓ is from $\ell = \ell_o + (mg/k)$ to $\ell \to \infty$. For $\ell \to \ell_o + (mg/k)$, we find $\theta \to 0$, $r \to 0$, $z \to \ell_o + (mg/k)$, and $\omega \to \omega_{min} = \sqrt{g/(\ell_o + mg/k)}$. At the opposite extreme, and we find $\theta \to \pi/2$ rad $= 90^o$, $r \to \ell$, $z \to mg/k$, and $\omega \to \omega_{max} = \sqrt{k/m}$.

6-25

(a) Referring to Fig. 6E-25b, the difference $D = 2y$.

(b) The volume of the water is $A\ell$, so the total mass $M_w = \rho A \ell$.

(c) The restoring force $F = -\rho g A D = -2\rho g A y$.

(d) Under the stated assumption, we have the equation

$$M_w \frac{d^2y}{dt^2} = F \qquad \text{or}$$

$$\rho A \ell \frac{d^2y}{dt^2} = -2\rho gAy$$

Canceling common factors, we obtain

$$\frac{d^2y}{dt^2} = \frac{-2gy}{\ell}$$

This is the equation of harmonic motion of angular frequency $\sqrt{2g/\ell}$ and period $\underline{2\pi \sqrt{\ell/2g}}$.

6-26

(a) We let d_o denote the equilibrium depth. In equilibrium, the buoyant force $Ad_o\rho g$ must exactly balance the weight mg. Therefore $Ad_o\rho g = mg$, or $\underline{d_o = m/\rho A}$.

(b) Letting the downward direction be positive, the net force is given by

$$F = -\rho Agd + mg = \underline{-\rho Ag(d - d_o)}$$

From this equation, we see that if the block is displaced downward $(d > d_o)$, the net force is upward $(F < 0)$. If the block is displaced upward $(d < d_o)$, the net force is downward $(F > 0)$. Therefore the equilibrium is stable.

(c) Assuming that we can neglect the fluid motion that accompanies the block's motion, we have

$$m \frac{d^2(d - d_o)}{dt^2} = -\rho Ag(d - d_o)$$

which is the equation of harmonic oscillations.

(d) Referring to the previous equation, we find that the angular frequency $\omega = \sqrt{\rho Ag/m}$, so the (ordinary) frequency $= \underline{(1/2\pi)(\sqrt{\rho Ag/m})}$.

6-27

The equation to be solved is

$$\frac{d^2x}{dt^2} = -\alpha x - \beta \frac{dx}{dt}$$

with the added information that $\alpha = \beta^2/4$, so that the equation becomes

$$\frac{d^2x}{dt^2} = \frac{-\beta^2 x}{4} - \beta\frac{dx}{dt}$$

The trial solution is given by

$$x_T = (A + Bt)\ e^{-\beta t/2}$$

where A and B are arbitrary constants. To check this solution we evaluate the first and second derivatives, obtaining

$$\frac{dx_T}{dt} = Be^{-\beta t/2} + (-\beta/2)(A + Bt)e^{-\beta t/2}$$

$$= [(B - \beta A/2) + (-\beta Bt/2)]e^{-\beta t/2}$$

and

$$\frac{d^2x_T}{dt^2} = (-\beta B/2)e^{-\ t/2} + (-\beta/2)[(B - \beta A/2) + (-\beta Bt/2)]e^{-\beta t/2}$$

$$= [(-\beta B + \beta^2 A/4) + (\beta^2 Bt/4)]e^{-\beta t/2}$$

We find that

$$-\frac{\beta^2 x_T}{4} - \beta\frac{dx_T}{dt} = (-\beta^2/4)(A + Bt)e^{-\beta t/2}$$

$$-\beta[(B - \beta A/2) + (-\beta Bt/2)]e^{-\beta t/2}$$

$$= [(-\beta^2/4 + \beta^2/2)A - \beta B + (-\beta^2 Bt/4 + \beta^2 Bt/2)]e^{-\beta t/2}$$

$$= [(-\beta B + \beta^2 A/4) + (\beta^2 Bt/4)]e^{-\beta t/2}$$

This is identical with d^2x_T/dt^2, which verifies the proposed solution.

6-28

The equation to be solved is

$$\frac{d^2x}{dt^2} = -\alpha x - \beta\frac{dx}{dt}$$

with $(\beta^2/4) > \alpha$. The trial solution to be verified is

$$x_T = \left(Ae^{\gamma t} + Be^{-\gamma t}\right)e^{-\beta t/2}$$

where $\gamma \equiv \sqrt{\beta^2/4 - \alpha}$. We evaluate the first and second derivatives of the trial solution, obtaining

$$\frac{dx_T}{dt} = (\gamma - \beta/2)Ae^{(\gamma - \beta/2)t} + (-\gamma - \beta/2)Be^{(-\gamma - \beta/2)t}$$

and

$$\frac{d^2x_T}{dt^2} = (\gamma - \beta/2)^2Ae^{(\gamma - \beta/2)t} + (-\gamma - \beta/2)^2Be^{(-\gamma - \beta/2)t}$$

Assembling the right-hand side of the basic equation, we find

$$-\alpha x_T - \beta \frac{dx_T}{dt} = -\alpha Ae^{(\gamma - \beta/2)t} - \alpha Be^{(-\gamma - \beta/2)t}$$

$$- \beta(\gamma - \beta/2)Ae^{(\gamma - \beta/2)t} - \beta(-\gamma - \beta/2)Be^{(-\gamma - \beta/2)t}$$

$$= [(-\alpha - \beta\gamma + \beta^2/2)Ae^{(\gamma - \beta/2)t} + (-\alpha + \beta\gamma + \beta^2/2)Be^{-(\gamma - \beta/2)t}$$

Comparing this expression with the expression for d^2x_T/dt^2, we see that the solution will be verified if

$$(\gamma - \beta/2)^2 = (-\alpha - \beta\gamma + \beta^2/2) \quad \text{and}$$

$$(-\gamma - \beta/2)^2 = (-\alpha + \beta\gamma + \beta^2/2)$$

In order for the first member of this pair of displayed equations to be valid, we must have

$$\gamma^2 - \beta\gamma + \beta^2/4 = -\alpha - \beta\gamma + \beta^2/2$$

or

$$\gamma^2 = \beta^2/4 - \alpha$$

which is precisely the <u>definition</u> of γ. The other member of the pair of equations yields the same requirement, so we have established the desired result.