$$g_1(u) = \frac{g_n(u_{n1}u)}{u_{n1}\left|[dg_n(\beta)/d\beta]\right|_{\beta = u_{n1}}} \qquad (5)$$

Equations (1) and (5) are explicit solutions for ν_1 and g_1 in terms of ν_n and g_n. A similar procedure can be followed for any k between 1 and n - 1. Here we give the results:

$$\nu_k = u_{nk}\nu_n \qquad (6)$$

$$g_k(u) = \frac{g_n(u_{nk}u)}{u_{nk}\left|dg_n(\beta)/d\beta\right|_{\beta = u_{nk}}} \qquad (7)$$

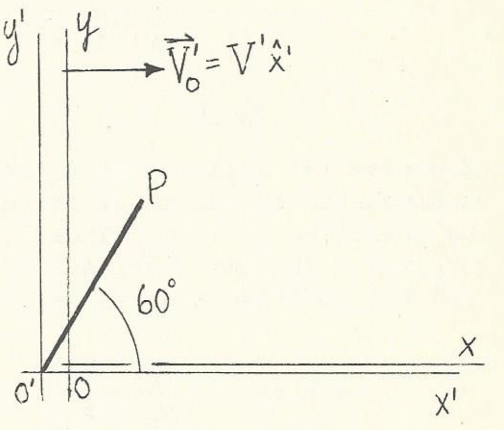
Notice that eqs. (6) - (7) are consistent with eqs. (1) and (5). Furthermore, we observe that eqs. (6) and (7) are also valid for n = k, in which case they reduce to mathematical identities. That is, eqs. (6) and (7) are valid for k = 1,2,3,...,n.

CHAPTER FOURTEEN

14-10

Using the conventional orientation and relative motion of the primed and unprimed frames, the Lorentz position-time transformation includes the equation $t' = \gamma(t - Vx/c^2)$. If we consider two events which, in the unprimed frame, happen in the same place ($\Delta x = 0$) and are separated by a time interval Δt, then Δt is the proper time. In the primed frame, the two events occur at times separated by $\Delta t' = \gamma(\Delta t - V\Delta x/c^2) = \gamma\Delta t = \Delta t/\sqrt{1 - V^2/c^2}$. This is the time-dilation formula.

14-11



The situation is indicated in the figure at the right. We notice that the observer O is moving to the right (in the positive-x' direction) at velocity V' = 0.8c with respect to observer O'.

(a) As measured in its rest frame (the primed frame), point P at the "upper" end of the meter stick has coordinates x'_P = (1 m) $\cos 60^\circ$ = 1/2 m and y'_P = (1 m) $\sin 60^\circ$ = $\sqrt{3}/2$ m.

Point Q at the "lower" end of the meter stick has coordinates $x'_Q = 0$, $y'_Q = 0$. For convenience, we suppose that observers O and O' both use the coincidence of their reference origins to determine the zero of time. Then the equations which relate space-time measurements made in O to those in O' are

$$x = \gamma'(x' - V't') \qquad (1)$$

$$y = y' \qquad (2)$$

$$z = z' \qquad (3)$$

$$t = \gamma'(t' - V'x'/c^2) \qquad (4)$$

where $\gamma' \equiv [1 - (V'^2/c^2)]^{-1/2}$. Since $V' = 0.8c$, $\gamma' = 5/3$. Equations (1) - (4) differ slightly in appearance from Eqs. (14-16). This is simply because the present exercise has O moving to the right in the frame of O', rather than the reverse. We need to determine the coordinates (x_Q,y_Q) and (x_P,y_P) of the ends of the meter stick at a given instant in the unprimed frame. Since the stick is moving to the left with speed 0.8c in the unprimed frame, it is crucial that the coordinates of Q and P be determined for the same time t. It is convenient to choose $t = 0$ as the instant for the measurements. The point Q has coordinates $(x'_Q,y'_Q) = (0,0)$, so eq. (4) tells us that $t'_Q = 0$ for the measurement of Q's position at $t = 0$. Then eq. (1), with $x'_Q = 0$ and $t'_Q = 0$ gives $x_Q = 0$. This should not surprise us, since we adopted $t = t' = 0$ when the primed and unprimed origins coincided. Equation (3) gives us $y_Q = y'_Q = 0$. Therefore we have $(x_Q,y_Q) = (0,0)$ at $t = 0$. Now we must evaluate (x_P,y_P) at $t = 0$. As before, eq. (3) gives y_P easily: $y_P = y'_P = \sqrt{3}/2$ m. Referring to eq. (4), with $t = 0$ and $x' = x'_P$, we have $t'_P = V'x'_P/c^2$. Then eq. (1), with $x' = x'_P$ and $t' = t'_P = V'x'_P/c^2$, we obtain

$$x_P = \gamma'[x'_P - V'(V'x'_P/c^2)]$$

$$= \frac{1}{\sqrt{1 - (V')^2/c^2}}[1 - (V')^2/c^2]x'_P$$

$$= x'_P/\gamma' \qquad (5)$$

Equation (5) expresses the "length contraction" suffered by the x-dimension of the meter stick. With $x'_P = 1/2$ m and $\gamma' = 5/3$, we obtain $x_P = 0.3$ m. Therefore the coordinates of P at $t = 0$ are $(x_P,y_P) = (0.3 \text{ m}, \sqrt{3}/2 \text{ m})$. Then the angle θ between the line QP and the positive x axis, as measured by observer O, is given by

$$\theta = \tan^{-1}\left(\frac{y_P - y_Q}{x_P - x_Q}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2 - 0}{3.0 - 0}\right)$$

$$\theta = \tan^{-1}(2.887) = \underline{70.9^{\circ}}$$

(b) The length ℓ of the meter stick is given by

$$\ell = \sqrt{(y_P - y_Q)^2 + (x_P - x_Q)^2}$$

$$= \sqrt{(\sqrt{3}/2)^2 + (0.3)^2} = \underline{0.917 \text{ m}}$$

14-12

(a) We assume that the strobe is at the origin of the primed frame, so that the light flash travels "out" along the meter stick from Q to P (as identified in the solution for Exercise 14-11). Then the light is traveling with speed c in a direction at 60° with respect to the x' axis. Therefore the velocity (u'_x, u'_y) of the flash is given by $(c \cos 60^{\circ}, c \sin 60^{\circ}) = \underline{(c/2, c\sqrt{3}/2)}$.

(b) We emphasize that as discussed in the solution for Exercise 14-11 the roles of the primed and unprimed frames are the reverse of their usual roles. Therefore the relevant velocity transformation equations are

$$u_x = \frac{u'_x - V'}{1 - V'u'_x/c^2} \qquad \text{and}$$

$$u_y = \frac{\sqrt{1 - (V')^2/c^2}}{1 - V'u'_x/c^2} u'_y$$

With $(u'_x, u'_y) = (c/2, c\sqrt{3}/2)$ and $V' = 4c/5$, we find

$$u_x = \frac{c/2 - 4c/5}{1 - (4c/5)(c/2)/c^2} = \frac{-0.3c}{0.6} = \underline{-c/2}$$

and

$$u_y = \frac{\sqrt{1 - (4c/5)^2/c^2}}{1 - (4c/5)(c/2)/c^2} \cdot (c\sqrt{3}/2)$$

$$= \frac{0.6}{0.6} \cdot (c\sqrt{3}/2) = \underline{c\sqrt{3}/2}$$

That is, in the unprimed frame the light flash has a velocity $(u_x, u_y) = \underline{(-c/2, c\sqrt{3}/2)}$.

(c) The path of the light flash in the unprimed frame lies at an angle α with respect to the positive x axis, where

$$= \tan^{-1}\left(\frac{u_y}{u_x}\right) = \tan^{-1}\left[\frac{(c\sqrt{3}/2)}{(-c/2)}\right]$$

$$= \tan^{-1}(-\sqrt{3}) = \underline{120^\circ}$$

In the primed frame, the direction of propagation of the light flash is parallel to the edge of the meter stick. However, in the unprimed frome, the angle of the path of the flash is not the same as the "tilt" angle of the meter stick (120° vs. 70.9°).

(d) In the unprimed frame, the speed of the light flash is given by $\sqrt{u_x^2 + u_y^2} = [(-c/2)^2 + (c\sqrt{3}/2)^2]^{1/2} = c$, as stated.

14-13

The Lorentz transformations appropriate for the frames described in Sec. 14-6 include the equations $x' = \gamma(x - Vt)$ and $y' = y$. Therefore if a point P is at rest in the primed frame, so that its coordinates are (x'_P, y'_P) for all times t', then the point's unprimed coordinates at time $t = 0$ will be $(x_P, y_P) = [(x'_P/\gamma), y'_P]$. For later times, $(x_P, y_P) = [(x'_P/\gamma) + Vt, y'_P]$.

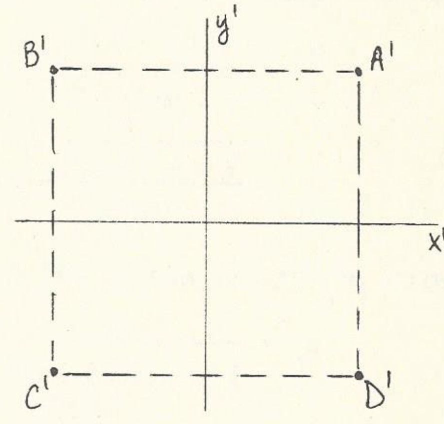
(a) The situation in the primed frame is shown in the top figure at the right. The coordinates of the corners are:

$(x'_A, y'_A) = (1/2\text{ m}, 1/2\text{ m})$

$(x'_B, y'_B) = (-1/2\text{ m}, 1/2\text{ m})$

$(x'_C, y'_C) = (-1/2\text{ m}, -1/2\text{ m})$

$(x'_D, y'_D) = (1/2\text{ m}, -1/2\text{ m})$.



The situation in the unprimed frame at the instant $t = 0$ is shown in the lower figure at the right. The coordinates of the corners are:

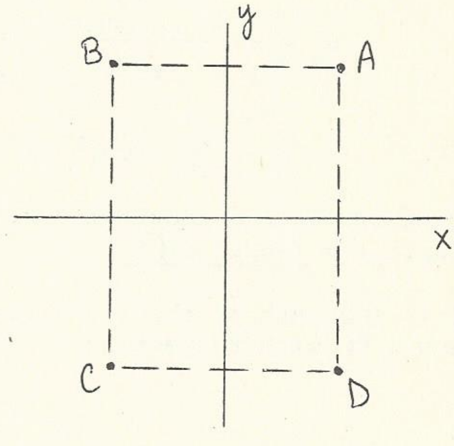
$(x_A, y_A) = [1/(2\gamma)\text{ m}, 1/2\text{ m})]$

$(x_B, y_B) = [-1/(2\gamma)\text{ m}, 1/2\text{ m}]$

$(x_C, y_C) = [-1/(2\gamma)\text{ m}, -1/2\text{ m}]$

$(x_D, y_D) = [1/(2\gamma)\text{ m}, -1/2\text{ m}]$

As measured in the unprimed frame, the slab is a rectangle of dimensions $1/\gamma$ m by 1 m.



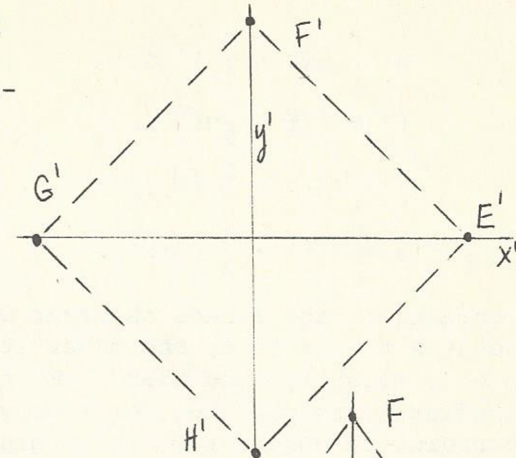
(b) The situation in the primed frame is shown in the upper figure at the right. The coordinates of the corners are:

$(x'_E y'_E) = (1/\sqrt{2}\ \text{m},\ 0)$

$(x'_F y'_F) = (0,\ 1/\sqrt{2}\ \text{m})$

$(x'_G y'_G) = (-1/\sqrt{2}\ \text{m},\ 0)$

$(x'_H, y'_H) = (0,\ -1/\sqrt{2}\ \text{m})$



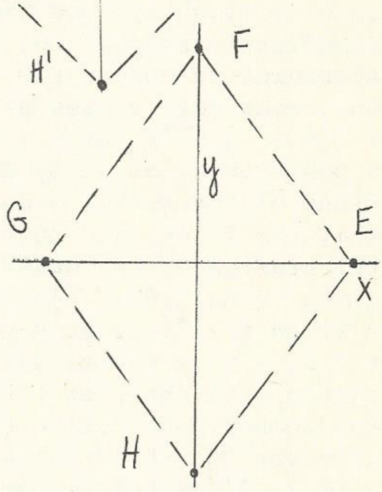
The situation in the unprimed frame at the instant t = 0 is shown in the lower figure at the right. The coordinates of the corners are:

$(x_E, y_E) = [1/(\gamma\sqrt{2})\ \text{m},\ 0]$

$(x_F, y_F) = (0,\ 1/\sqrt{2}\ \text{m})$

$(x_G, y_G) = [-1/(\gamma\sqrt{2})\ \text{m},\ 0]$

$(x_H, y_H) = (0,\ -1/\sqrt{2}\ \text{m})$



As measured in the unprimed frame, the slab is a "diamond" of total width $(\sqrt{2}/\gamma)$ m in x and total height $\sqrt{2}$ m in y.

14-14

Since the rocket ship of proper length L_o = 100 m is Lorentz-contracted to L = 50 m, its speed V can be found from $L = L_o/\gamma$. We find $\gamma = 2$, so that $V = c\sqrt{3}/2 = 2.60 \times 10^8$ m/s. We let frame O be fixed in the tunnel, with its origin at the center, with the front gate at x_1 = 25 m, and with the rear gate at x_2 = -25 m. We let frame O' be fixed in the rocket ship, with its origin at the center of the ship, with the nose of the ship at x'_3 = 50 m, and with the tail of the ship at x'_4 = -50 m. We suppose that the origins of O and O' coincide at t = t' = 0. Then the relevant Lorentz transformation equations are

$$x' = \gamma(x - Vt) \qquad \text{and} \qquad (1)$$

$$t' = \gamma(t - Vx/c^2) \qquad (2)$$

The inverse transformations are

$$x = \gamma(x' + Vt') \qquad (3)$$

$$t = \gamma(t' + Vx'/c^2) \qquad (4)$$

With the numerical values $\gamma = 2$ and $V = c\sqrt{3}/2$ inserted, eqs. (1) - (4) become

$$x' = 2(x - \frac{1}{2}\sqrt{3}\,ct) \qquad (5)$$

$$t' = 2(t - \frac{1}{2}\sqrt{3}\,x/c) \qquad (6)$$

$$x = 2(x' + \frac{1}{2}\sqrt{3}\,ct') \qquad (7)$$

$$t = 2(t' + \frac{1}{2}\sqrt{3}\,x'/c) \qquad (8)$$

According to the tunnel observer O, both gates drop at time $t = 0$, one at $x = x_1 = 25$ m, the other at $x = x_2 = -25$ m. Since the rocket is only slightly less than 50 m in length, the front gate must immediately reopen, but for a very brief time both gates are closed, according to observer O. According to the rocket-borne observer O', the front gate closes at a time t_1' given by Eq. (6): $t_1' = 2(0 - \frac{1}{2}\sqrt{3}\,x_1/c) = -(\sqrt{3})(25)/(3.00 \times 10^8) = -1.44 \times 10^{-7}$ s, at a position x_1' given by Eq. (5): $x_1' = 2(x_1 - 0) = 50$ m, just in front of the rocket's nose. Observer O' agrees with observer O that the front gate must (and does) immediately reopen. As for the rear gate, the observer O' finds that it closes at a time t_2' given by Eq. (6): $t_2' = 2(0 - \frac{1}{2}\sqrt{3}\,x_2/c) = -(\sqrt{3})(-25)/(3.00 \times 10^8) = +1.44 \times 10^{-7}$ s, at a position x_2' given by Eq. (5): $x_2' = 2(x_2 - 0) = -50$ m, just behind the rocket's tail. Observer O' agrees with observer O that the rear gate could not have been closed any sooner. We notice that $t_2' > t_1'$: that is, according to observer O', the front gate closes (and then immediately reopens) 2.88×10^{-7} s <u>before</u> the rear gate closes. Observer O' states that at no instant are both gates closed with the ship between them. This differs from the conclusion of observer O, for whom the two gates are both closed (very briefly) at $t = 0$ with the ship between them. Although the descriptions of observers O and O' differ dramatically, both of the descriptions are correct. At first encounter, it is very difficult for most (if not all) physics students to believe that there are no algebraic mistakes, logical inconsistencies, or other loopholes in this analysis. The reason for the difficulty is simple: although the analysis is entirely correct, the conclusion is one that is completely foreign to an intuition based on the Newtonian concept of absolute simultaneity.

<u>14-15</u>

(a) For convenience we assume that $t = t' = 0$ when the origins of the primed and unprimed frames coincide. Carefully noticing that O is moving to the right with respect to O', we find the transformation equations

$$x = \gamma(x' - Vt') \qquad \text{and} \qquad (1)$$

$$t = \gamma(t' - Vx'/c^2) \qquad (2)$$

where $\gamma \equiv (1 - V^2/c^2)^{-1/2}$.

We are informed that, in the primed frame, event 1 occurs at $x_1' = -L'/2$ and $t_1' = L'/2c$, while event 2 occurs at $x_2' = L'/2$ and $t_2' = t_1' = L'/2c$. Using eq. (2) we find that in the unprimed frame event 1 occurs at time t_1 given by

$$t_1 = \gamma[(L'/2c) - V(-L'/2)/c^2]$$

$$= (\gamma L'/2c)(1 + V/c)$$

Event 2 occurs at time t_2 given by

$$t_2 = \gamma[(L'/2c) - V(L'/2)/c^2]$$

$$= (\gamma L'/2c)(1 - V/c)$$

Therefore $t_2 - t_1 = (\gamma L'/2c)(-2V/c) = \underline{-\gamma L'V/c^2}$. Event 2 occurs before event 1 in the unprimed frame.

(b) Equation (1) implies that

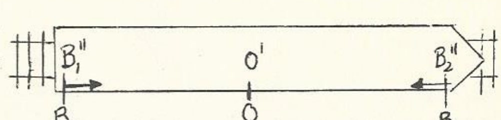
$$\Delta x \equiv x_2 - x_1 = \gamma(\Delta x' - V\Delta t')$$

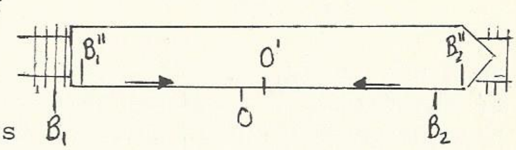
where $\Delta x' \equiv x_2' - x_1'$ and $\Delta t' \equiv t_2' - t_1'$. For the given events, $\Delta x' = L'$ and $\Delta t' = 0$. Therefore $\Delta x = \underline{\gamma L'}$.

(c) The spatial separation Δx found in part (b) does not represent the length of an object because the positions x_1 and x_2 were not measured simultaneously ($t_2 \neq t_1$).

14-16

As in the text discussion, we assume that the blasting caps are at rest in the unprimed frame at positions B_1 and B_2, which are equidistant from O, with B_2 "down the track" from O, and B_1 "back up the track" from O. However, in this experiment we suppose that the caps are detonated simultaneously as judged in the reference frame of O', just as O and O' pass one another. The upper and lower figures at right show two successive illustrations of the situation based on measurements made in the primed reference frame. The upper one shows the situation at the instant of detonation, while the lower one shows the situation at a slightly later time. The short arrows represent the light flashes produced by the detonation. Notice that when the blasting caps explode they are located at B_1'' and B_2'' in the primed frame.





(These are not coincident with the points B_1' and B_2' mentioned in the text. That is because the experiments are different experiments. Below we shall see other consequences of the difference in the experiments.) As indicated in the lower figure on the previous page, the observer O' will afterwards know that the detonations were conducted as planned because the two light flashes will arrive simultaneously at O', and because the observer can measure the distances $O'B_1''$, and $O'B_2''$ and will find them to be equal. (We note here that $O'B_1'' = O'B_2''$ will not be the same as the unprimed observer's distances $OB_1 = OB_2$ which are the same in both experiments since the blasting caps are located at fixed positions in the ground. We also repeat that $O'B_1'' = O'B_2''$ will not be the same as the distances $O'B_1' = O'B_2'$ measured by the primed observer in the experiment discussed in the text. In fact, $O'B_1'' > O'B_1'$. The ratio of the two distances can be evaluated with the help of the Lorentz transformation.)

As can be seen from the second figure above, the light flashes do not arrive together at observer O: the flash from blasting cap #1 arrives first. Observer O' interprets this as being due to the motion of observer O toward the rear of the train and away from the front of the train. Observer O agrees that he/she receives the flash from cap #1 before the flash from cap #2, but does not accept the explanation of observer O'. Since the distances OB_1 and OB_2 are equal, observer O must conclude that cap #1 was detonated before cap #2 was detonated. As in the text, the two observers disagree about the relative timing of spatially separated events. In the experiment analyzed here, observer O correctly concludes that the cap at the rear of the train was detonated some time before the cap at the front, even though observer O' concludes (also correctly) that the two detonations were simultaneous. Both observers are correct: the simultaneity (or lack of it) for spatially separated events is an observer-dependent property. Simultaneity is "relative", not "absolute".

14-17

The role of the quantity δ is indicated in Eq. (14-11):

$$t' = \gamma(t + \delta) \qquad (1)$$

We need to construct a physical (rather than purely algebraic) explanation of the functional form

$$\delta = -Vx/c^2 \qquad (2)$$

obtained in Eq. (14-13). In doing so, we put aside consideration of the factor γ that appears on the right side of eq. (1). The thought experiment of Sec. 14-4 considers two events (1 and 2) that occur at different locations ($x_1 < x_2$) but at the same instant ($t_1 = t_2$) in an inertial frame (the unprimed frame). The results of the thought experiment indicate that, as observed by another inertial observer O' moving with speed V in the positive x direction, the events 1 and 2 are not simultaneous ($t'_1 \neq t'_2$). Specifically, the observer O' finds that event 2, which occurred at the larger x-value, happened before event 1. That is, with $x_1 < x_2$ and $t_1 = t_2$, we found $t'_1 > t'_2$. As discussed in Sec. 14-4, from the point of view of the unprimed observer, the lack of simultaneity of the events as measured by O' is due to the motion of O' toward the location of event 2 and away from the location of event 1. Therefore, the effect [to first order in V, ignoring γ in Eq. (1)] is based upon a difference in transit times, in a manner reminiscent of the Newtonian Doppler effect. As can be seen from Fig. 14-8, the differential effect on transit times, caused by the motion of observer O', depends on the distance $\Delta x = x_2 - x_1$ between the event locations in the unprimed frame, and on the speed V of observer O'. This differential effect should be proportional to Δx, should have $\Delta t'$ to be negative if $\Delta x > 0$, and should (for fairly low speeds V) be proportional to V. This last feature can be understood by carefully examining Fig. 14-8. The necessary features are exactly expressed by eqs. (1) and (2). Applying eq. (1) to events 1 and 2 with $\Delta t \equiv t_2 - t_1 = 0$, $\Delta x \equiv x_2 - x_1 \neq 0$, $\Delta t' \equiv t'_2 - t'_1$, and with δ given by eq. (2), we find

$$\begin{aligned}\Delta t' &= \gamma[\Delta t + \Delta\delta] \\ &= \gamma\left[0 - \Delta\left(\frac{Vx}{c^2}\right)\right] \\ &= \frac{-\gamma V \Delta x}{c^2}\end{aligned}$$

This result, which is demanded by the thought experiment of Sec. 14-4, depends upon the particular form of δ given in eq. (2). The quantity δ is necessarily linear in x, linear in V, and contains the sign factor -1.

14-18

The roles of the quantity γ are indicated in Eqs. (14-11):

$$x' = \gamma(x - Vt) \tag{1}$$

$$t' = \gamma(t + \delta) \tag{2}$$

We need to construct a physical explanation for the dependence of γ on V, as given by Eq. (14-15):

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} \tag{3}$$

Within Sec. 14-5, Figures 14-9 and 14-10, and the discussion leading up to Eq. (14-3) are particularly relevant here. We notice from Fig. 14-9 that in eq. (1) we should use $V = -|V|$, since the positive x direction is taken to be rightward (as is conventional). The time dilation thought experiment shows that the time interval $T' = 2\Delta t'$ between the departure of the light pulse from clocks C_1' and C and its return to clocks C_2' and C, as reckoned by observer O', is longer than the time interval $T = 2\Delta t$ measured by O. As summarized in Eq. (14-3), this results from the fact that in the unprimed frame the light need only travel a total distance 2ℓ, while in the primed frame the light travels twice the length of the hypotenuse of a triangle whose sides are ℓ' and $|V|\Delta t'$. With $\ell' = \ell$, as is shown in Fig. 14-11 and its caption, the quantity Δt given by

$$\Delta t = \ell/c \qquad (4)$$

while $\Delta t'$ satisfies the equation

$$\begin{aligned} c^2(\Delta t')^2 &= \ell'^2 + |V|^2(\Delta t')^2 \\ &= \ell^2 + |V|^2(\Delta t')^2 \end{aligned} \qquad (5)$$

Using (4), the solution of (5) is

$$\Delta t' = \Delta t \frac{1}{\sqrt{1 - V^2/c^2}} \qquad (6)$$

The Lorentz position-time transformation given in eq. (2), with $\delta = -Vx/c^2$ (as established in Exercise 14-17, must be consistent with the result of the time-dilation thought experiment as summarized in eq. (6). It is easily verified that γ must therefore have the form given in eq. (3). Applying eq. (2) to the departure and then to the return of the light signals, and taking the difference of the resulting equations, we have

$$\begin{aligned} 2\Delta t' \equiv T' &= \gamma T + \frac{\gamma V(x_f - x_i)}{c^2} \\ &= \gamma T \equiv \gamma(2\Delta t) \end{aligned} \qquad (7)$$

since $x_f = x_i$. (The clock C is fixed in the unprimed frame.) Equation (7) is consistent with eq. (6) only if γ is as given in eq. (3). To summarize in less algebraic and more physical terms, the particular dependence of γ on $|V|$ and (more evidently) its lack of dependence on the sign of V are implied by the geometry of the time-dilation thought experiment of Sec. 14-4.

14-19

(a) We display Equations (14-16) here for later reference:

$$x' = \gamma(x - Vt) \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

$$t' = \gamma(t - Vx/c^2) \tag{4}$$

These equations enable an observer O to calculate the event coordinates (x',y',z't') that would be measured in a (primed) coordinate system moving with velocity $V\hat{x}$. Now observer O' is also an inertial observer, and he/she observes that O is moving with velocity $V'\hat{x}' = -V\hat{x}'$. Because the form of the transformations, as given by eqs. (1) - (4), must be the same for all inertial observers, the observer O' must be able to obtain the equations giving (x,y,z,t) in terms of (x',y',z',t') simply by replacing V with V' = -V in eqs. (1) - (4) and simultaneously interchanging primed and unprimed quantities. Therefore we obtain

$$x = \gamma'(x' - V't) = \gamma(x' + Vt') \tag{5}$$

$$y = y' \tag{6}$$

$$z = z' \tag{7}$$

$$t = \gamma'(t' - V'x'/c^2) = \gamma(t' + Vx'/c^2) \tag{8}$$

where we have used the fact that

$$\gamma' \equiv \frac{1}{\sqrt{1 - (V')^2/c^2}} = \frac{1}{\sqrt{1 - (-V)^2/c^2}}$$

$$= \frac{1}{\sqrt{1 - V^2/c^2}} = \gamma$$

(b) We need to solve eqs. (1) and (4) for x and t in terms of x' and t'. From eq. (1), we have

$$x = \frac{x'}{\gamma} + Vt \tag{9}$$

and from eq. (4), we have

$$t = \frac{t'}{\gamma} + \frac{Vx}{c^2} \tag{10}$$

Substituting eq. (10) into eq. (9), we find

$$x = \frac{x'}{\gamma} + \frac{Vt'}{\gamma} + \frac{V^2x^2}{c^2}$$

Solving this for x, we find

$$x = \frac{1}{(1 - V^2/c^2)}\left(\frac{x' + Vt'}{\gamma}\right) = \gamma(x' + Vt') \qquad (11)$$

Substituting eq. (11) into eq. (10), we obtain

$$t = \frac{t'}{\gamma} + \frac{\gamma Vx'}{c^2} + \frac{\gamma V^2 t'}{c^2}$$

$$= \gamma\left[t'\left(\frac{1}{\gamma^2} + \frac{V^2}{c^2}\right) + \frac{Vx'}{c^2}\right]$$

$$= \gamma(t' + Vx'/c^2) \qquad (12)$$

Equations (11) and (12) agree with eqs. (5) and (8); the inversion of the y' and z' equations is trivial.

14-20

We note that the exercise statement includes the tacit assumption that $\gamma' = \gamma$ when V' = -V.

(a) At time t', the light-sphere observed by O' intersects the positive x' axis at a coordinate x' = ct'. Therefore x'/t' = c.

(b) At time t, the light-sphere observed by O intersects the positive x axis at a coordinate x = ct. Therefore x/t = c.

(c) Since the equations obtained in parts (a) and (b) refer to the same point-event, we use x' = ct' and x = ct in Eqs. (14E-1) and (14E-2):

$$ct' = \gamma(ct - Vt) \quad \text{and} \qquad (1)$$

$$ct = \gamma(ct' + Vt') \qquad (2)$$

Multiplying eqs. (1) and (2) we have

$$c^2 tt' = \gamma^2(c^2 - V^2)tt'$$

Solving this for γ, we find

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} \qquad (3)$$

(d) When eq. (3) is used in Eqs. (14E-1), and (14E-2), we obtain

$$x' = \frac{1}{\sqrt{1 - V^2/c^2}}(x - Vt) \qquad (4)$$

and

$$x = \frac{1}{\sqrt{1 - V^2/c^2}} (x' + Vt') \qquad (5)$$

These equations relate arbitrary event-coordinate observations of observers O and O'. Using eq. (4) to eliminate x' from eq. (5), we find

$$x = \frac{(x - Vt)}{(1 - V^2/c^2)} + \frac{Vt'}{\sqrt{1 - V^2/c^2}}$$

Solving this for t', we find (after some algebra)

$$\underline{t' = \frac{1}{\sqrt{1 - V^2/c^2}} \left(t - \frac{Vx}{c^2} \right)}$$

which is the standard Lorentz transformation equation for t'.

<u>14-21</u>

(a) The interval $(t_1 - 0)$ is a <u>proper</u> time because it is determined by a single clock at rest in the frame of reference fixed to the earth.

(b) The spaceship's clock reads an <u>improper</u> time interval since the two events occur at different places in the spaceship's rest frame. Using the Lorentz time transformation, the spaceship's clock reads $t_1' = \gamma(t_1 - v\, 0/c^2) = \underline{\gamma t_1}$.

(c) The velocity of the earth relative to the spaceship is -v. Therefore, as reckoned by the spaceship, the earth has receded to a distance given by $|-v| t_1' = \underline{\gamma v t_1}$.

(d) The interval $(t_2 - 0)$ is <u>improper</u> because the events occur at different places in the earth's frame of reference.

(e) The following events occur at the same place in the spaceship's frame of reference. At time t' = 0, the earth passes the spaceship. At a later time $t = t_2'$, the super-spaceship passes the spaceship. Using the differenced Lorentz transformation $\Delta t = \gamma(\Delta t' + v\Delta x'/c^2)$ with $\Delta x' = 0$, we have $\Delta t = \gamma \Delta t'$. But $\Delta t' = t_2' - 0 = t_2'$ while $\Delta t = t_2 - 0 = t_2$. Therefore $t_2' = \underline{t_2/\gamma}$. The interval $(t_2' - 0)$ is a <u>proper</u> time interval.

(f) Combining the results of parts (b) and (e), the "pursuit" required an elapsed time $T_p' \equiv t_2' - t_1' = (t_2/\gamma) - \gamma t_1$, as reckoned in the spaceship's reference frame.

(g) In the spaceship's reference frame, the super-spaceship had to travel the distance γvt_1 found in part (c).

(h) Combining the results of parts (f) and (g), we find the velocity V' of the fast ship, as reckoned by the slow one:

$$V' = \frac{\gamma vt_1}{(t_2/\gamma) - \gamma t_1} \qquad (1)$$

As shown in the statement of the exercise, the time t_2 is given by

$$t_2 = \frac{Vt_1}{(V - v)} \qquad (2)$$

Combininb eqs. (2) and (3), we obtain

$$V' = \frac{\gamma vt_1}{\left\{\left[\frac{Vt_1}{\gamma(V - v)}\right] - \gamma t_1\right\}} = \frac{(V - v)v}{\left[\frac{V}{\gamma^2} - (V - v)\right]}$$

Since $1/\gamma^2 = 1 - v^2/c^2$, this reduces to

$$V' = \frac{(V - v)v}{\left(-\frac{Vv^2}{c^2} + v\right)} = \frac{V - v}{1 - Vv/c^2}$$

With the proper attention to differences in notation, this is found to agree with the Lorentz velocity transformation given in the text.

14-22

We let the primed frame be the reference frame fixed to the water. Then the Lorentz velocity transformation relating the lab velocity v_x to the velocity v_x' in the primed frame is

$$v_x = \frac{v_x' + V}{1 + \frac{v_x' V}{c^2}} \qquad (1)$$

(Here we have made the usual choice that the primed frame is moving in the direction of increasing x.) For a flash of light traveling in the same direction as the water flow, we have

$$v_x' = \frac{c}{n} \qquad (2)$$

(We must realize that the "speed of light in still water" is to be read as "the speed of light with respect to the water".) According to equations (1) and (2) the velocity in the laboratory is given by

$$v_x = \frac{\frac{c}{n} + |V|}{1 + \frac{c|V|}{nc^2}}$$

Since $|V| \ll c$, this can be expanded and approximated:

$$v_x = \left(\frac{c}{n} + |V|\right) \cdot \left[1 - \frac{|V|}{nc} + \left(\frac{|V|}{nc}\right)^2 - \dots \right]$$

$$= \frac{c}{n} + |V| - \frac{|V|}{n^2} - \frac{|V|^2}{nc} + \frac{|V|^2}{n^3 c} + \dots$$

$$\simeq \frac{c}{n} + |V| \cdot \left(1 - \frac{1}{n^2}\right)$$

<u>14-23</u>

(a) Since we are assuming that the speed of light with respect to the ether is c and that the Galilean transformations apply, when the light is traveling from P to M_2, its speed is <u>$c - V$</u> relative to the apparatus.

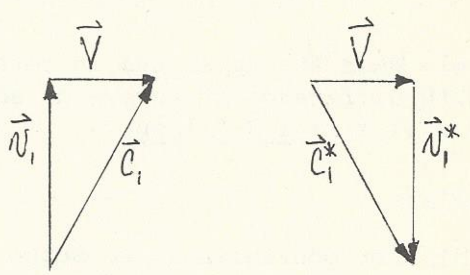
(b) The time t_A required for the light to travel from P to M_2 is given by $t_A =$ <u>$L/(c - V)$</u>.

(c) When the light is traveling from M_2 back to P, its speed is <u>$c + V$</u> relative to the apparatus.

(d) The time t_B required for the return trip from M_2 to P is given by $t_B =$ <u>$L/(c + V)$</u>. The time t_2 required for the entire trip (P to M_2 to P) is given by

$$t_2 = t_A + t_B = \frac{L}{c - V} + \frac{L}{c + V} = \underline{\frac{2Lc}{c^2 - V^2}}$$

(e) The situation is shown in the (Galilean) velocity addition diagrams at the right. The light reaching M_1 from P must have a velocity vector $\vec{v}_1$, with respect to the apparatus. Because the apparatus is moving through the ether with velocity $\vec{V}$, the velocity $\vec{v}_1 = \vec{c}_1 - \vec{V}$, where $\vec{c}_1$ is the velocity of the light with respect to the ether. Because $|\vec{c}_1| = c$ and $\vec{v}_1$ is perpendicular to $\vec{V}$, the magnitude of $\vec{v}_1$ is given by $|\vec{v}_1| = \sqrt{c^2 - V^2}$. Similarly,



the light traveling from M_1 back to P must have a velocity vector $\vec{v}_1^* = \vec{c}_1^* - \vec{V}$ with respect to the apparatus. Again we have $|\vec{c}_1^*| = c$ and $\vec{v}_1^* \perp \vec{V}$. Therefore $|\vec{v}_1^*| = |\vec{v}_1| = \sqrt{c^2 - V^2}$. The speed of the light relative to the apparatus is $\underline{\sqrt{c^2 - V^2}}$ on both parts of the round trip from P to M_1 to P.

(f) Since the total distance traveled is 2L, the round-trip time is given by $t_1 = \underline{2L/\sqrt{c^2 - V^2}}$.

(g) The difference in round trip times is given by

$$t_2 - t_1 = \frac{2Lc}{c^2 - V^2} - \frac{2L}{\sqrt{c^2 - V^2}}$$

$$= \frac{2L}{c}\left[\left(1 - \frac{V^2}{c^2}\right)^{-1} - \left(1 - \frac{V^2}{c^2}\right)^{-1/2}\right]$$

$$= \frac{2L}{c}\left[\left(1 + \frac{V^2}{c^2} + \dots\right) - \left(1 + \frac{1}{2}\frac{V^2}{c^2} + \dots\right)\right]$$

$$\simeq \frac{LV^2}{c^3}$$

as claimed.

(h) The time difference $t_2 - t_1$ corresponds to a path difference $d = c(t_2 - t_1) = \underline{LV^2/c^2}$.

(i) The path difference $c(t_2 - t_1)$ is equal to $\underline{(LV^2/\lambda c^2)}$ wave-lengths.

(j) With $L = 10$ m, $\lambda = 6 \times 10^{-7}$ m, and $V = 3 \times 10^4$ m/s, we find

$$\frac{d}{\lambda} = \frac{10}{6 \times 10^{-7}} \frac{(3 \times 10^4)^2}{(3 \times 10^8)^2} = \frac{1}{6}$$

(k) When the apparatus is rotated through 90°, the sign of the path difference is reversed, so the fringe shift should be $2 \times (1/6) = \underline{1/3 \text{ fringe}}$.

14-24

(a) For convenience, we suppose that the hoop is centered on the origin of the primed system. The equation that describes the circle is therefore

$$(x')^2 + (y')^2 = (a')^2 \qquad (1)$$

If the circle is observed in the unprimed frame at time t, the Lorentz transformations imply that $x' = \gamma(x - Vt)$ and $y' = y$. Therefore eq. (1) becomes

$$\gamma^2(x - Vt)^2 + y^2 = (a')^2$$

or

$$\frac{(x - Vt)^2}{(a'/\gamma)^2} + \frac{y^2}{(a')^2} = 1 \qquad (2)$$

This is a standard form for the equation of an ellipse centered at $x_c = Vt$ and $y_c = 0$, with semimajor axis $a = a'$ parallel to the y axis and with semiminor axis $b = a'/\gamma = a'\sqrt{1 - V^2/c^2}$ along the x axis.

(b) Applying the definition of the eccentricity and the result of part (a), we have

$$e = \sqrt{1 - (a')^2(\sqrt{1 - V^2/c^2})^2/(a')^2}$$

$$= \sqrt{1 - \left(1 - \frac{V^2}{c^2}\right)} = \frac{|V|}{c}$$

Therefore for the given speeds, the eccentricity values are: (1) 0.010, (2) 0.10, (3) 0.50, (4) 0.90, (5) 0.999.

14-25

(a) The period of the flash would be increased by a factor $\gamma \equiv 1/\sqrt{1 - |v_s|^2/c^2}$, so the period would be $\gamma/\nu = \gamma$ microseconds.

(b) Since the light travels at speed c, the earthbound observer will reckon that a light flash travels a distance $c\cdot(\gamma/\nu) = \gamma c/\nu$ in each interval between flashes.

(c) Since the spaceship is traveling at speed $|v_s|$, the earthbound observer will reckon that the ship travels a distance $|v_s|\cdot(\gamma/\nu) = \gamma|v_s|/\nu$ between successive flashes.

(d) Using the results of parts (b) and (c), as the ship travels toward the earth, the earthbound observer finds that the successive flashes are separated by a distance $D = \gamma c/\nu - \gamma|v_s|/\nu = (\gamma/\nu)\cdot(c - |v_s|)$.

(e) The time interval between the arrival of successive flashes, as reckoned on the earth, is simply the distance D divided by c.

$$T = D/c = (\gamma/\nu)(1 - |v_s|/c) \qquad \text{or}$$

$$T = \frac{1}{\sqrt{1 - |v_s|^2/c^2}} \frac{(1 - |v_s|/c)}{\nu} = \frac{1}{\nu}\sqrt{\frac{1 - |v_s|/c}{1 + |v_s|/c}}$$

Since $1/\nu = 1$ microsecond $\equiv 1\ \mu s$, $T = \sqrt{(1 - |v_s|/c)/(1 + |v_s|/c)}\ \mu s$.

(f) With $|v_s| = 3c/5$, $T = \sqrt{(1 - 3/5)/(1 + 3/5)}\ \mu s = \sqrt{2/8}\ \mu s$ $= \underline{0.5\ \mu s}$; the corresponding frequency $\nu' = 1/T = \underline{2\text{ MHz}}$.

(g) By looking over the solution for parts (a) - (e), we see that the correct result can be obtained simply be replacing $|v_s|$ with $-|v_s|$ wherever it appears. Therefore we find that the period would be $\sqrt{(1 + |v_s|/c)/(1 - |v_s|/c)}$ if the source is moving away from the earth at speed $|v_s|$. For the particular speed examined in part (f), we find a received period of $\underline{2\ \mu s}$ and a received frequency of $\underline{0.5\text{ MHz}}$.

(h) In general, for a source of intrinsic frequency ν moving at speed $|v_s|$ either directly toward or directly away from the earth, the received frequency ν' is given by

$$\nu' = \nu\sqrt{\frac{1 \pm |v_s|/c}{1 \mp |v_s|/c}}$$

where the upper signs apply to cases in which the source is approaching, and the lower signs apply to cases in which the source is receding.

<u>14-26</u>

Examining the relativistic Doppler effect formula obtained in part (h) of Exercise 14-25, we observe that the observed frequency ν' is smaller than the emitted frequency ν if and only if the source is receding. We are given $\nu = 8.0 \times 10^{14}$ Hz and $\nu' = 5.0 \times 10^{14}$ Hz, so $\nu'/\nu < 1$ and we conclude that the source is receding. The speed $|v_s|$ is determined by

$$\frac{\nu'}{\nu} = \sqrt{\frac{1 - |v_s|/c}{1 + |v_s|/c}} \qquad (1)$$

Solving eq. (1) for $(|v_s|/c)$, we obtain

$$\frac{|v_s|}{c} = \frac{1 - (\nu'/\nu)^2}{1 + (\nu'/\nu)^2} \qquad (2)$$

With $\nu'/\nu = 5/8$, we find $|v_s|/c = [1 - (5/8)^2]/[1 + (5/8)^2]$ $= (39/64)/(89/64) = 0.438$. The source is <u>receding at a speed of 0.438c</u>. (We have assumed purely radial motion.)

14-27

(a) As stated in the solution for part (h) of Exercise 14-25, if v is the speed of a source relative to an observer and if ν_o is the intrinsic frequency of the source, then the observer will receive signals at frequency

$$\nu_a = \nu_o\sqrt{\frac{1 + v/c}{1 - v/c}} \qquad (1)$$

as the source approaches. The received frequency as the source recedes is given by

$$\nu_r = \nu_o\sqrt{\frac{1 - v/c}{1 + v/c}} \qquad (2)$$

In the present case, ν_a and ν_r are known, while v and ν_o are to be determined. Dividing eq. (2) by eq. (1), we find

$$\frac{\nu_r}{\nu_a} = \frac{1 - v/c}{1 + v/c}$$

Solving this for v/c, we obtain

$$\frac{v}{c} = \frac{1 - (\nu_r/\nu_a)}{1 + (\nu_r/\nu_a)} \qquad (3)$$

With the given numerical values ν_a = 130 MHz and ν_r = 60 MHz, eq. (3) yields v/c = (1 - 6/13)/(1 + 6/13) = 7/19, so v = 7c/19 = <u>0.368c</u>.

(b) Multiplying eq. (2) by eq. (1), we find $\nu_a\nu_r = \nu_o^2$. Therefore $\nu_o = \sqrt{\nu_a\nu_r} = \sqrt{(60)(130)}$ = <u>88.3 MHz</u>.

(c) Reading the assignment carefully, we conclude that we are asked to find received frequency of those signals which are <u>emitted</u> by the probe <u>at the time of minimum separation</u> of probe and ship. (These signals are not received until sufficient time elapses for the light to propagate across that minimum separation distance.) For these particular signals, the wavefronts are not "pushed together" (as they are when the ship-to-probe distance is decreasing), nor are they "pulled apart" (as they are when the ship-to-probe is increasing). The only factor which operates to change the frequency is time dilation. (Cf. part (a) of Exercise 14-25.) Therefore the frequency ν^* at which those signals are received is given by $\nu^* = \nu_o/\gamma = \nu_o\sqrt{1 - v^2/c^2}$. With ν_o = 88.3 MHz and v/c = 7c/19, we find ν^* = <u>82.1 MHz</u>.

14-28

(a) Adopting the conventional choice of coordinate axes, the x, x', and x'' axes are parallel to the relative motions, frame O" moves with velocity $V'\hat{x}'$ with respect to frame O' and the frame O' moves with velocity $V\hat{x}$ with respect to frame O. The equations for the Lorentz transformation from O to O' are

$$x' = \gamma(x - Vt) \qquad (1)$$

$$y' = y \qquad (2)$$

$$z' = z \qquad (3)$$

$$t' = \gamma(t - Vx/c^2) \qquad (4)$$

where $\gamma \equiv 1/\sqrt{1 - V^2/c^2}$. The equations for the Lorentz transformation from O' to O" are

$$x'' = \gamma'(x' - V't) \qquad (5)$$

$$y'' = y' \qquad (6)$$

$$z'' = z' \qquad (7)$$

$$t'' = \gamma'(t' - V'x'/c^2) \qquad (8)$$

where $\gamma' \equiv 1/\sqrt{1 - (V')^2/c^2}$. Using eqs. (1) and (4) to eliminate x' ant t' from eqs. (5) and (8), we find

$$x'' = \gamma'[\gamma(x - Vt) - V'\gamma(t - Vx/c^2)]$$

$$t'' = \gamma'[\gamma(t - Vx/c^2) - V'\gamma(x - Vt)/c^2]$$

or

$$x'' = \gamma\gamma'\cdot\left(1 + \frac{VV'}{c^2}\right)\cdot x - t\cdot\gamma\gamma'(V + V') \qquad (9)$$

$$t'' = \gamma\gamma'\cdot\left(1 + \frac{VV'}{c^2}\right)\cdot t - \frac{x}{c^2}\cdot\gamma\gamma'(V + V') \qquad (10)$$

Equations (9) and (10), plus the easily obtained equations $y'' = y$ and $z'' = z$, constitute the desired equations giving (x'', y'', z'', t'') in terms of (x, y, z, t).

(b) Referring to eqs. (9) and (10), we see that if we define

$$\gamma'' \equiv \gamma\gamma'(1 + VV'/c^2) \qquad (11)$$

and

$$V'' \quad \frac{V + V'}{(1 + VV'/c^2)} \qquad (12)$$

the transformation from O to O" has the standard form

$$x'' = \gamma''(x - V''t) \qquad (13)$$

$$y'' = y \qquad (14)$$

$$z'' = z \qquad (15)$$

$$t'' = \gamma''(t - V''x/c^2) \qquad (16)$$

However, we must verify that the definitions given in eqs. (11) and (12) are consistent. That is, we must show that γ'' $= 1/\sqrt{1 - (V'')^2/c^2}$. We make this confirmation by showing that $1/(\gamma'')^2$ equals $1 - (V'')^2/c^2$. From eq. (11), we have

$$\frac{1}{(\gamma'')^2} = \frac{1}{\gamma^2}\frac{1}{\gamma'^2}\frac{1}{(1 + VV'/c^2)^2}$$

$$= \frac{(1 - V^2/c^2)[1 - (V')^2/c^2]}{(1 + VV'/c^2)^2}$$

$$= \frac{1 - [V^2 + (V')^2]/c^2 + V^2(V')^2/c^4}{(1 + VV'/c^2)^2}$$

$$= \frac{1 + 2VV'/c^2 + V^2(V')^2/c^4 - [V^2 + 2VV' + (V')^2]/c^2}{(1 + VV'/c^2)^2}$$

$$= 1 - \frac{1}{c^2}\left[\frac{V + V'}{(1 + VV'/c^2)}\right]^2$$

$$= 1 - \left(\frac{V''}{c}\right)^2$$

where the last step uses eq. (12). This establishes the consistency of eqs. (11) and (12). Therefore the equations (13) - (16) constitute a Lorentz transformation from frame O to frame O".

(c) The velocity of O" with respect to O' is $v'_x = V'$, and the velocity of O' with respect to O is V. If we apply the Lorentz velocity transformation given in the text, with proper attention to the notation, we conclude that the velocity v_x of O" with respect to O should be given by

$$v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}} = \frac{V' + V}{1 + \frac{V'V}{c^2}}$$

This is identical with the quantity V" which characterizes the transformation from O to O", as found in part (b).

(d) We have shown by direct calculation that if frames O and O' are related by the Lorentz transformation eqs. (1) - (4), and if O' and O" are related by the Lorentz transformation eqs. (5) - (8), then O and O" are related by the Lorentz transformation eqs. (13) - (16). The velocity V" characterizing the transformation from O to O" is precisely the velocity of O" in frame O, as given by the Lorentz velocity transformation.

(e) The table is given below, with the one column for each meter stick and one row for each reference frame. Each entry is the length (in meters) assigned to "that column's meter stick" by "that row's observer".

Meter Sticks

Observers	M	M'	M"
O	1	$1/\gamma$	$1/\gamma''$
O'	$1/\gamma$	1	$1/\gamma'$
O"	$1/\gamma''$	$1/\gamma'$	1

14-29

We let frame O_1 be at rest with respect to frame O and also have its axes parallel to those of O. However, we let the coordinate origin of O_1 be located at the point (x_o, y_o, z_o) in frame O, and we let time measurements in O_1 have their zero when the time is t_o in frame O. Then for any point event, the coordinates in O_1 are related to those in O by

$$x_1 = x - x_o \quad (1)$$

$$y_1 = y - y_o \quad (2)$$

$$z_1 = z - z_o \quad (3)$$

$$t_1 = t - t_o \quad (4)$$

Equations (1) - (4) constitute the transformation from O to O_1 Similarly, we let frame O'_1 be at rest with respect to frame O' and also have its axes parallel to those of O'. We let the coordinate origin of O_1' be located at the point (x_o', y_o', z_o') in frame O', and we let time measurements in O_1' have their zero when the time is t_o' in frame O'. The transformation from O' to O_1' is given by

$$x_1' = x' - x_o' \qquad (5)$$

$$y_1' = y' - y_o' \qquad (6)$$

$$z_1' = z' - z_o' \qquad (7)$$

$$t_1' = t' - t_o' \qquad (8)$$

Since frame O' moves with velocity $V\hat{x}$ with respect to frame O, the frame O_1' moves with velocity $V\hat{x}$ with respect to frame O_1. Furthermore, since the point-event (x_o, y_o, z_o, t_o) in frame O coincides with the point event (x_o', y_o', z_o', t_o') in frame O', eqs. (1) - (8) show that the point event (0,0,0,0) in frame O_1 coincides with the point event (0,0,0,0) in frame O_1'. Therefore the transformation from frame O_1 to frame O_1' is the standard Lorentz transformation:

$$x_1' = \gamma(x_1 - Vt_1) \qquad (9)$$

$$y_1' = y_1 \qquad (10)$$

$$z_1' = z_1 \qquad (11)$$

$$t_1' = \gamma(t_1 - Vx_1/c^2) \qquad (12)$$

where $\gamma \equiv 1/\sqrt{1 - V^2/c^2}$. By using eqs. (1) - (8) in eqs. (9) - (12), we obtain the desired equations that express the transformation from frame O to frame O':

$$x' - x_o' = \gamma[(x - x_o) - V(t - t_o)] \qquad (13)$$

$$y' - y_o' = y - y_o \qquad (14)$$

$$z' - z_o' = z - z_o \qquad (15)$$

$$t' - t_o' = \gamma[(t - t_o) - V(x - x_o)/c^2] \qquad (16)$$

14-30

[Note to instructors: We acknowledge here that the transformed wave equation obtained in this problem is not a perfect "O' - analogue" of the original equation in frame O, since the quantity being differentiated is the field f (as measured in O) rather than the field f' (as measured in O'). However, in the instructive and commonly-considered case of plane electromagnetic waves, both the electric and magnetic fields satisfy the given wave equation in frame O. Furthermore, under the Lorentz transformation appropriate when the relative velocity of frames O and O' is along the x axis (that is, along the direction of propagation of the plane waves), the electric and magnetic fields measured by O' at (x',t') are linear combinations of the electric and magnetic fields measured by O at the same point-event (x,t). Under these circumstances, by showing that f (as measured in O, but regarded as a function of x' and t') satisfies the wave equation in O' whenever it satisfies the wave equation in O, we actually do establish that f' (the field as measured in O') will satisfy the wave equation in O' whenever f satisfies it in O. More generally, Exercise 14-30 can be regarded as establishing the form-invariance of the operator $(\partial^2/\partial x^2) - (1/c^2)(\partial^2/\partial t^2)$.]

We wish to re-express the wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \qquad (1)$$

as a differential equation in the variables (x',t') rather than (x,t) where

$$x' = \gamma(x - Vt) \qquad \text{and} \qquad (2)$$

$$t' = \gamma(t - Vx/c^2) \qquad (3)$$

with $\gamma \equiv 1/\sqrt{1 - V^2/c^2}$. By the chain rule, the first derivatives of f are

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x} \qquad (4)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial t} \qquad (5)$$

From eqs. (2) and (3), we find $\partial x'/\partial x = \gamma$, $\partial t'/\partial x = -\gamma V/c^2$, $\partial x'/\partial t = -\gamma V$, and $\partial t'/\partial t = \gamma$. Using these in eqs. (4) and (5), we find

$$\frac{\partial f}{\partial x} = \gamma\frac{\partial f}{\partial x'} - \gamma\frac{V}{c^2}\frac{\partial f}{\partial t'} \qquad (6)$$

$$\frac{\partial f}{\partial t} = -\gamma V\frac{\partial f}{\partial x'} + \gamma\frac{\partial f}{\partial t'} = \gamma\frac{\partial f}{\partial t'} - \gamma V\frac{\partial f}{\partial x'} \qquad (7)$$

Differentiating eq. (6) with respect to x, using the chain rule, and recognizing that γ, V, and c are constants, we obtain

$$\frac{\partial^2 f}{\partial x^2} = \gamma\left[\frac{\partial}{\partial x}\frac{\partial f}{\partial x'}\right] - \frac{\gamma V}{c^2}\left[\frac{\partial}{\partial x}\frac{\partial f}{\partial t'}\right]$$

....

$$= \gamma\left[\frac{\partial^2 f}{\partial (x')^2}\frac{\partial x'}{\partial x} + \frac{\partial^2 f}{\partial t'\partial x'}\frac{\partial t'}{\partial x}\right]$$

$$- \frac{\gamma v}{c^2}\left[\frac{\partial^2 f}{\partial x'\partial t'}\frac{\partial x'}{\partial x} + \frac{\partial^2 f}{\partial (t')^2}\frac{\partial t'}{\partial x}\right]$$

$$= \gamma^2 \frac{\partial^2 f}{\partial (x')^2} - \frac{2\gamma^2 v}{c^2}\frac{\partial^2 f}{\partial x'\partial t'} + \frac{\gamma^2 v^2}{c^2}\frac{\partial^2 f}{\partial (t')^2} \qquad (8)$$

Similarly, differentiating eq. (7) with respect to t, we find

$$\frac{\partial^2 f}{\partial t^2} = \gamma\left[\frac{\partial}{\partial t}\frac{\partial f}{\partial t'}\right] - \gamma v\left[\frac{\partial}{\partial t}\frac{\partial f}{\partial x'}\right]$$

$$= \gamma\left[\frac{\partial^2 f}{\partial (t')^2}\frac{\partial t'}{\partial t} + \frac{\partial^2 f}{\partial x'\partial t'}\frac{\partial x'}{\partial t}\right]$$

$$- \gamma v\left[\frac{\partial^2 f}{\partial t'\partial x'}\frac{\partial t'}{\partial t} + \frac{\partial^2 f}{\partial (x')^2}\frac{\partial x'}{\partial t}\right]$$

$$= \gamma^2 \frac{\partial^2 f}{\partial (t')^2} - 2\gamma^2 v\frac{\partial^2 f}{\partial x'\partial t'} + \gamma^2 v^2 \frac{\partial^2 f}{\partial (x')^2} \qquad (9)$$

Using eqs. (8) and (10) we have

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \gamma^2\left(1 - \frac{v^2}{c^2}\right)\cdot\left[\frac{\partial^2 f}{\partial (x')^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial (t')^2}\right]$$

$$= \frac{\partial^2 f}{\partial (x')^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial (t')^2} \qquad (10)$$

Equation (10) shows that when (x,t) and (x',t') are related by eqs. (2) and (3), the differential operator $[(\partial^2/\partial (x')^2] - (1/c^2)[\partial^2/\partial (t')^2]$ is identical to the differential operator $(\partial^2/\partial x^2) - (1/c^2)(\partial^2/\partial t^2)$. In particular, if f(x,t) obeys eq. (1), then the dependent variable f, now regarded as a function of x' and t', obeys the equation

$$\frac{\partial^2 f}{\partial (x')^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial (t')^2} = 0$$

with the same wave speed: c' = c. The speed of the waves is the same in both frames. This result is in constrast to the result obtained when the wave equation is transformed using the Galilean transformation ($x' = x - Vt$; $t' = t$). In that case, the predicted wave velocities in O' are $c_r' = c - V$ (for waves traveling to the right in O) and $c_\ell' = -c - V$ (for waves traveling to the left in O).

14-31

(a) Let event 1 be the arrival of the protons at the earth (at time t_1 in the earth's frame). Let event 2 be the emission of a radio burst by Jupiter: this event occurs at Jupiter at time $t_2 = t_1 + \Delta t$. The spatial separation of the events is $\Delta x \equiv x_2 - x_1$. The transformation of time intervals from the earth frame (O) to the rocket frame (O') is given by

$$\Delta t' = \gamma(\Delta t - |V|\Delta x/c^2) \qquad (1)$$

With the given numerical values $\Delta t = 900$ s, $\Delta x = 6.3 \times 10^8$ km, and $|V| = 0.50$ c, we find $\gamma = 2/\sqrt{3}$, and

$$\Delta t' = \frac{2}{\sqrt{3}}\left[900 - \frac{0.5(6.3 \times 10^{11})}{(3 \times 10^8)}\right]$$

$$= -173 \text{ s}$$

This means that, in the rocket frame, the radio burst was emitted at Jupiter before the outbound protons passed the earth. Therefore the protons could not have caused the radio burst in the rocket's frame (or in any other frame).

(b) The two events are simultaneous in a frame traveling with velocity V* if

$$\Delta t^* = \gamma^*(\Delta t - V^*\Delta x/c^2) = 0 \qquad (2)$$

With the given numerical values for Δx and Δt, we find

$$\frac{V^*}{c} = \frac{c\Delta t}{\Delta x} = \frac{(3 \times 10^8 \text{ m/s})(900 \text{ s})}{(6.3 \times 10^{11} \text{ m})} = \frac{3}{7}$$

which is realizable. Events 1 and 2 are simultaneous in a frame of reference traveling from the earth toward Jupiter at speed 3c/7.

(c) For any burst that the protons could trigger, the time interval between events 1 and 2 must be nonnegative in every frame of reference. Referring to eqs. (1) and (2), we see that this requires that $\Delta t - (V\Delta x/c^2)$ be a positive quantity for all $V < c$. This requires Δt to equal or exceed $\Delta x/c$. Any burst that the protons trigger must be emitted at least $(6.3 \times 10^{11}$ m$)/(3 \times 10^8$ m/s$)$

$= 2.1 \times 10^3$ s after the protons pass the earth. That is, $\Delta t \geq \Delta t_{min} = 2100 \text{ s} = 35.0 \text{ min}$.

(d) Proper time intervals are intervals between events that occur at a given location. Suppose that two events are separated spatially by $\Delta x \equiv x_2 - x_1$, and occur a time interval $\Delta t \equiv t_2 - t_1$, apart in one inertial frame (the unprimed frame). Then the inertial observer O' who measures a proper time interval between them is the one who travels with constant velocity V from location x_1 at time t_1 to location x_2 at time t_2. With the given numerical values, we have $V = (\Delta x/\Delta t) = (6.3 \times 10^{11} \text{ m})/(3600 \text{ s}) = 1.75 \times 10^8 \text{ m/s} = \underline{0.583c}$. Then $\sqrt{1 - V^2/c^2} = 0.812$ and the proper time $\Delta t' = \Delta t\sqrt{1 - V^2/c^2} = (3600 \text{ s})(0.812) = 2.92 \times 10^3 \text{ s} = \underline{48.7 \text{ min}}$. Any inertial observer O" who is moving with respect to observer O' will measure a time interval $\Delta t''$ that exceeds $\Delta t'$.

14-32

(a) According to the problem statement, the particle's velocity in frame O' is given by

$$\vec{v}' \equiv v'_x\hat{x}' + v'_y\hat{y}' = V\cos\theta'\hat{x}' + V\sin\theta'\hat{y}' \qquad (1)$$

where V is also the speed of relative motion of frames O and O'. The particle's velocity in frame O is given by

$$\vec{v} \equiv v_x\hat{x} + v_y\hat{y} = V\cos\theta\hat{x} + V\sin\theta\hat{y} \qquad (2)$$

Applying the Lorentz velocity transformation given in Eqs. (14-21), we find that

$$v'_x = V\cos\theta' = \frac{v_x - V}{1 - v_xV/c^2} = \frac{V(\cos\theta - 1)}{1 - (V^2\cos\theta/c^2)} \qquad (3)$$

and

$$v'_y = V\sin\theta' = \frac{\sqrt{1 - V^2/c^2}\cdot v_y}{1 - v_xV/c^2} = \frac{V\sqrt{1 - V^2/c^2}\,\sin\theta}{1 - (V^2\cos\theta/c^2)} \qquad (4)$$

Because the speed v of the particle with respect to O equals the speed v' of the particle with respect to O', we can conclude that the observed angle θ between v and x should equal the angle $(180^\circ - \theta')$ between v' and x'. (This conclusion is based on the symmetry properties of the Lorentz transformation, which imply that $\theta' = \theta$ when $v' = v$, even for cases in which the common speed $v' = v$ is <u>not</u> equal to V, the relative speed of O and O'. If such considerations do not persuade you that θ' and θ must be supplementary angles, eqs. (3) and (4) can be employed to prove the result analytically.) With $\theta = 180^\circ - \theta'$, we have $\cos\theta' = -\cos\theta$ and $\sin\theta' = \sin\theta$. Then eq. (4) suffices, yielding

$$1 = \frac{\sqrt{1 - V^2/c^2}}{1 - (V^2 \cos\theta/c^2)}$$

so that

$$\cos\theta = \frac{1 - \sqrt{1 - V^2/c^2}}{V^2/c^2} = \frac{1}{1 + \sqrt{1 - V^2/c^2}} \qquad (5)$$

We finally obtain

$$\theta = \cos^{-1}\left(\frac{1}{1 + \sqrt{1 - V^2/c^2}}\right) \qquad (6)$$

and

$$\theta' = 180^\circ - \theta \qquad (7)$$

Equations (6) and (7) are easily shown to give reasonable results in two limiting cases. As $V \to 0$, $\theta \to \cos^{-1}(1/2) = 60^\circ$, and $\theta' \to 120^\circ$. As $V \to c$, $\theta \to \cos^{-1}(1) = 0^\circ$, and $\theta' \to 180^\circ$.

(b) The results follow immediately from eqs. (6) and (7).

V/c	θ	$\theta' = 180^\circ - \theta$
0.1	59.9°	120.1°
0.3	59.2°	120.8°
0.9	45.9°	134.1°
0.99	28.8°	151.2°

14-33

(a) The problem statement implicitly assumes that frames O and O' are related by the standard form of the Lorentz transformation, as given in Eqs. (14-16). The inverse transformation is given by

$$x = \gamma(x' + Vt') \qquad (1)$$

$$y = y' \qquad (2)$$

$$z = z' \qquad (3)$$

$$t = \gamma(t' + Vx'/c^2) \qquad (4)$$

We apply eqs. (1) - (4) to events 1 and 2 and then find the difference of the resulting equations for each coordinate:

$$x_2 - x_1 = \gamma[(x_2' - x_1') + V(t_2' - t_1')] \qquad (5)$$

$$y_2 - y_1 = y_2' - y_1' \tag{6}$$

$$z_2 - z_1 = z_2' - z_1' \tag{7}$$

$$t_2 - t_1 = \gamma[(t_2' - t_1') + V(x_2' - x_1')/c^2] \tag{8}$$

Using eqs. (5) - (8), we find

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

$$= \gamma^2[c(t_2' - t_1') + V(x_2' - x_1')/c]^2$$

$$- \gamma^2[(x_2' - x_1') + V(t_2' - t_1')]^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2$$

$$= \gamma^2[c^2(t_2' - t_1')^2 + 2V(x_2' - x_1')(t_2' - t_1') + V^2(x_2' - x_1')^2/c^2]$$

$$- \gamma^2[(x_2' - x_1')^2 + 2V(x_2' - x_1')(t_2' - t_1') + V^2(t_2' - t_1')^2]$$

$$- (y_2' - y_1')^2 - (z_2' - z_1')^2$$

$$= \gamma^2(c^2 - V^2)(t_2' - t_1')^2 - \gamma^2(1 - V^2/c^2)(x_2' - x_1')^2$$

$$- (y_2' - y_1')^2 - (z_2' - z_1')^2 \tag{9}$$

Since $\gamma^2(1 - V^2/c^2) = 1$, eq. (9) implies that

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

$$= c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2 \tag{10}$$

The quantity $c^2(\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]$ is invariant under the Lorentz transformation.

(b) The proper time interval between any two events is the time interval between them as measured in a reference frame in which the two events have the same spatial location. That is, in a frame O* in which $x_2^* = x_1^*$, $y_2^* = y_1^*$, and $z_2^* = z_1^*$, the elapsed $\Delta t^* \equiv t_2^* - t_1^*$ is the proper time interval. If we evaluate the invariant quantity of part (a) in the frame O*, we obtain $c^2(\Delta t^*)^2 - [(0)^2 + (0)^2 + (0)^2] = (\Delta t^*)^2$, as desired.

14-34

Using the Lorentz velocity transformation given in Eqs. (14-21), we find the square of the speed in frame O':

$$(v')^2 = (v'_x)^2 + (v'_y)^2 + (v'_z)^2$$

$$= \left(\frac{v_x - V}{1 - Vv_x/c^2}\right)^2 + \left(\frac{\sqrt{1 - V^2/c^2}\cdot v_y}{1 - Vv_x/c^2}\right)^2 + \left(\frac{\sqrt{1 - V^2/c^2}\cdot v_z}{1 - Vv_x/c^2}\right)^2$$

$$= \frac{1}{(1 - Vv_x/c^2)^2}\left[(v_x - V)^2 + (1 - V^2/c^2)(v_y^2 + v_z^2)\right]$$

$$= (1 - Vv_x/c^2)^{-2}[v_x^2 - 2v_xV + V^2 + v_y^2 + v_z^2 - V(v_y^2 + v_z^2)/c^2]$$

Since $v_x^2 + v_y^2 + v_z^2 = c^2$, we have

$$(v')^2 = (1 - Vv_x/c^2)^{-2}\left\{c^2 - 2v_xV + V^2[1 - (v_y^2 + v_z^2)/c^2]\right\}$$

But v = c implies that $1 - (v_y^2 + v_z^2)/c^2 = v_x^2/c^2$. Therefore we obtain

$$(v')^2 = (1 - Vv_x/c^2)^{-2}(c^2 - 2v_xV + V^2v_x^2/c^2)$$

$$= (1 - Vv_x/c^2)^{-2}c^2[1 - (2Vv_x/c^2) + (Vv_x/c^2)^2]$$

$$= (1 - Vv_x/c^2)^{-2}\, c^2\, (1 - Vv_x/c^2)^2 = c^2$$

That is, $\underline{v' = c}$: the speed of light is the same in all inertial frames.

14-35

(a) In the frame O, the velocity of the starlight is given by $\vec{v} = v_x\hat{x} + v_y\hat{y} = -c\hat{y}$. Using the Lorentz velocity transformation given in Eqs. (14-21), the velocity of the starlight in frame O' is $\vec{v}' = v'_x\hat{x}' + v'_y\hat{y}'$, where

$$v'_x = \frac{v_x - V}{1 - Vv_x/c^2} = -V \qquad (1)$$

and

$$v'_y = \frac{v_y\sqrt{1 - V^2/c^2}}{\left(1 - Vv_x/c^2\right)} = -c\sqrt{1 - V^2/c^2} \tag{2}$$

As observed from the earth, the angle $(\pi/2 - \varphi)$ between the starlight's velocity $\vec{v}'$ and the vector $-\hat{x}'$ satisfies the equation

$$\cos(\pi/2 - \varphi) = \frac{\vec{v}'\cdot(-\hat{x}')}{|\vec{v}'|\cdot|(-\hat{x}')|} = \frac{-v'_x}{v'} \tag{3}$$

Using eqs. (1) and (2), we find

$$v' = \sqrt{(v'_x)^2 + (v'_y)^2} = [V^2 + c^2(1 - V^2/c^2)]^{1/2} = c \tag{4}$$

Then eq. (3) yields $\sin\varphi = -(-V)/c = V/c$, as desired.

(b) The shift in the star's apparent direction is forward -- that is, in the same direction as the earth's motion with respect to the sun. This is indicated by the negative value of the x-component of the velocity $\vec{v}'$ of the incoming starlight.

(c) It was shown in part (a) that $v' = c$, as expressed in eq. (4).

(d) With $V = 3.00 \times 10^4$ m/s, $V/c = 10^{-4}$, so that $\varphi = \sin^{-1}(10^{-4}) = 10^{-4}$ radians. Since 1 radian $= (180^\circ/\pi) = (180/\pi)(60)^2$ seconds of arc $= 2.06 \times 10^5$ seconds of arc, φ = 20.6 arc seconds.

14-36

In order to relate $a' \equiv dv'/dt'$ to $a \equiv dv/dt$, we will need to use the Lorentz velocity transformation

$$v' = \frac{v - V}{(1 - vV/c^2)} \tag{1}$$

We omit the subscripts x on the velocities v and v' (and on the accelerations a and a') because the problem posed is essentially one-dimensional. According to the chain rule

$$a' \equiv \frac{dv'}{dt'} = \frac{dv'}{dv}\frac{dv}{dt}\frac{dt}{dt'}$$

$$= \frac{(dv'/dv)a}{(dt'/dt)} \tag{2}$$

Using eq. (1), we find

$$\frac{dv'}{dv} = \frac{(1)(1 - vV/c^2) - (-V/c^2)(v - V)}{(1 - vV/c^2)^2} = \frac{(1 - V^2/c^2)}{(1 - vV/c^2)^2} \tag{3}$$

To evaluate dt'/dt we need to begin with the Lorentz transformation equation for the time:

$$t' = \gamma(t - Vx/c^2) \qquad (4)$$

We find

$$\frac{dt'}{dt} = \gamma\left[1 - \left(\frac{V}{c^2}\right)\cdot\left(\frac{dx}{dt}\right)\right] = \gamma(1 - Vv/c^2) \qquad (5)$$

Using eqs. (3) and (5) in eq. (2), we obtain the desired result:

$$a' = \frac{(1 - V^2/c^2)}{(1 - vV/c^2)^2}\,\frac{a}{\gamma(1 - Vv/c^2)}$$

$$= \frac{(1 - V^2/c^2)^{3/2}}{(1 - vV/c^2)^3}\,a$$

CHAPTER FIFTEEN

15-16

As stated in Eq. (15-21a) and graphically represented in Fig. 15-10, the particle's instantaneous momentum $\vec{p}(t)$ and instantaneous energy E(t) are related by

$$E^2 = c^2\vec{p}\cdot\vec{p} + (m_oc^2)^2 \qquad (1)$$

Differentiating eq. (1) with respect to time, we find

$$2E\frac{dE}{dt} = 2c^2\vec{p}\cdot\frac{d\vec{p}}{dt}$$

or

$$\frac{dE}{dt} = \frac{c^2\vec{p}}{E}\cdot\frac{d\vec{p}}{dt} \qquad (2)$$

But $\vec{p} = m\vec{v} = \gamma m_o\vec{v}$ and $E = mc^2 = \gamma m_oc^2$, so $c^2\vec{p}/E = \vec{v}$. Using this in eq. (2), we obtain

$$\frac{dE}{dt} = \frac{d\vec{p}}{dt}\cdot\vec{v} = \vec{F}\cdot\vec{v} \qquad (3)$$

since $\vec{F} = d\vec{p}/dt$.