Hence $U/E = (3.8 \times 10^{17})/(6.1 \times 10^{18}) = 0.062 \simeq \underline{6\%}$.

(c) Since the earth rotates very nearly as a solid body, its rotational kinetic energy $K = \frac{1}{2} I \omega^2$, where I is the moment of inertia about its spin axis, and ω is the angular speed of its rotation. We have

$$\omega = 2\pi \frac{\text{radians}}{\text{day}} = \frac{2\pi \text{ rad}}{8.64 \times 10^4 \text{ s}}$$

$$= 7.27 \times 10^{-5} \text{ rad/s}$$

According to Exercise 10-39, the earth's moment of inertia $I = 8.04 \times 10^{37}$ kg·m^2, which is about 20% smaller than the value $2MR^2/5$ appropriate for a homogeneous sphere. Hence we find

$$K = \frac{1}{2}(8.04 \times 10^{37})(7.27 \times 10^{-5})^2$$

$$= 2.12 \times 10^{29} \text{ J} \simeq \underline{2.1 \times 10^{29} \text{ J}}$$

Hence $U/K = (3.8 \times 10^{17})/(2.12 \times 10^{29}) = 1.8 \times 10^{-12} \simeq \underline{2 \times 10^{-12}}$.

CHAPTER TWENTY-SIX

26-3

We use the symbols $\mathcal{M}$, L, T, and Q to represent the fundamental dimensions of mass, length, time, and electric charge, respectively. We use the symbol [x] to represent "the dimensions of the quantity x". Resistance is measured in volts/ampere, where 1 volt = 1 J/C and 1 A = 1 C/s. Hence 1 Ω = 1 J·s/C^2. But 1 J = 1 kg·m^2/s^2, so

$$[R] = (\mathcal{M}LT^{-2})(T)Q^{-2} = \mathcal{M}L^2T^{-1}Q^{-2} \qquad (1)$$

Capacitance is measured in coulombs/volt, or (coulombs)^2/joule. Therefore the dimensions of capacitance are

$$[C] = Q^2(\mathcal{M}L^2T^{-2})^{-1} = \mathcal{M}^{-1}L^{-2}T^2Q^2 \qquad (2)$$

Using eqs. (1) and (2), we find that

$$[RC] = [R][C] = (\mathcal{M}L^2T^{-1}Q^{-2})(\mathcal{M}^{-1}L^{-2}T^2Q^2) = T$$

which was to be shown.

26-4

We use the symbols $\mathcal{M}$, L, T, and Q to represent the fundamental dimensions of mass, length, time, and electric charge, respectively. We use the symbol [x] to represent "the dimensions of the quantity x". Inductance is measured in henries, where 1 H = 1 volt/(A/s) = 1 (J/C)/(C/s^2) = 1 J·s^2/C^2. Since 1 J = 1 kg·m^2/s^2, we obtain

$$[L] = (\mathcal{M}L^2T^{-2})(T^2)(Q^{-2}) = \mathcal{M}L^2Q^{-2} \qquad (1)$$

Capacitance is measured in coulombs/volt, or (coulombs)^2/joule. Therefore the dimensions of capacitance are

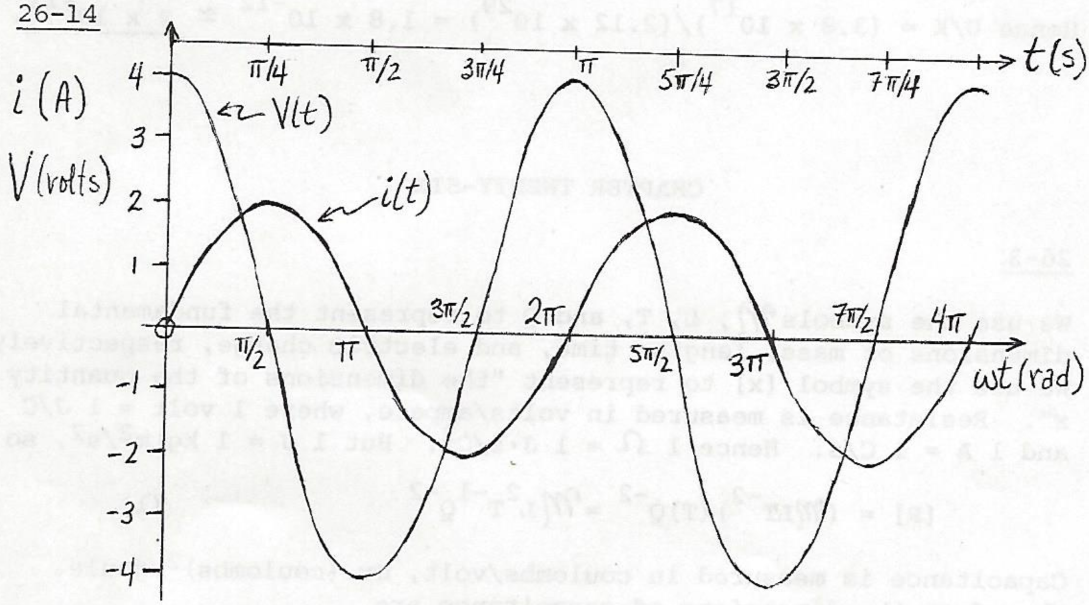
$$[C] = Q^2(\mathcal{M}L^2T^{-2})^{-1} = \mathcal{M}^{-1}L^{-2}T^2Q^2 \qquad (2)$$

Using eqs. (1) and (2), we find that

$$[1/\sqrt{LC}] = [L]^{-1/2}[C]^{-1/2} = (\mathcal{M}^{-1/2}L^{-1}Q)(\mathcal{M}^{1/2}LT^{-1}Q^{-1}) = T^{-1}$$

which are the dimensions of angular frequency.

26-14



The above graph displays the given current $i(t) = 2\sin[(2s^{-1})t]$ A and the driving voltage, which is given by

$$V(t) = L\frac{di}{dt} = (1\text{ H})(4\text{ A}\cdot s^{-1})\cos[(2s^{-1})t]$$

$$= 4\cos[(2s^{-1})t]\text{ V} \qquad (1)$$

The instantaneous power expenditure is given by

$$P(t) = i(t)V(t) = 8\sin[(2s^{-1})t]\cos[(2s^{-1})t]\text{ W} \quad \ldots\ldots$$

$= 4 \sin[(4s^{-1})t]$ W (2)

This is the rate at which the current delivers energy to the inductor. Equation (2) shows that the average value of P(t) is zero over any whole number of half-cycles. This can also be seen from the graph.

(I) For $0\ s < t < \pi/4$ s, i(t) and V(t) are both positive, so P(t) is positive. Energy is being supplied to the inductor's magnetic field, which is strongest when $|i(t)|$ is a maximum.

(II) For $\pi/4\ s < t < \pi/2$ s, i(t) is positive but V(t) is negative, so P(t) is negative. Energy is being drained from the inductor's magnetic field, which weakens as i(t) approaches zero. It can be seen from the graph that $|P(t)|$ runs through the same values and has the same average in interval II as in interval I. Since the algebraic signs of P(t) are opposite in the two intervals, the time-averaged power is zero over the entire time interval $0\ s \leq t \leq \pi/2$ s.

(III) For $\pi/2\ s < t < 3\pi/4$ s, i(t) and V(t) are both negative, so P(t) is positive. As in interval I, energy is being supplied to the strengthening magnetic field of the inductor.

(IV) For $3\pi/4\ s < t < \pi$ s, i(t) is negative but V(t) is positive, so P(t) is negative. Energy is being drained from the inductor as $|i(t)|$ approaches zero. As with intervals I and II, it can be seen from the graph that the average power is zero over the entire interval $\pi/2\ s \leq t \leq \pi$ s. In an AC circuit, the average power consumed by a pure inductor is zero: as the current oscillates, energy is alternately invested in and removed from the inductor's magnetic field.

26-16

As shown in Sec. 26-3 and summarized in Eq. (26-9), the current in the circuit of Fig. (26-3) is given by

$$i(t) = \frac{V_o}{R} e^{-(R/L)t} \equiv i\, e^{-(R/L)t}$$

after the switch is thrown from A to B at t = 0. Energy is dissipated into Joule heat by the resistor at a rate

$$P(t) = Ri^2(t) = i^2 R e^{-2(R/L)t}$$

The total energy consumed by the resistor after the switch is thrown is given by

$$E = \int_0^{\infty} P(t)dt = i^2 R \int_0^{\infty} e^{-2(R/L)t} dt$$ ......

$$= i^2 R \left[-\frac{L}{2R} e^{-2(R/L)t} \Big|_0^\infty \right]$$

$$= i^2 R \left(-\frac{L}{2R} \right) (e^{-\infty} - e^{-0}) = \left(-\frac{1}{2} Li^2 \right) (-1) = \frac{1}{2} Li^2$$

Here $i \equiv V_o/R$ is the current flowing when the switch is thrown.

26-17

Since the voltage source has been connected to the circuit for a long time prior to $t = 0$, the capacitor is fully charged and there is zero current when the switch is thrown. That is, at $t = 0$, the left plate of the capacitor in Fig. 26-5 has charge

$$q_o = CV_o \tag{1}$$

Once the switch is thrown to position B, Kirchoff's voltage law for the circuit is:

$$0 - V_R - V_C = 0$$

or

$$i(t) \cdot R + \frac{q(t)}{C} = 0$$

Referring to Fig. 26-5, we can see that $i = dq/dt$, so we obtain

$$R \frac{dq}{dt} + \frac{q}{C} = 0 \tag{2}$$

Equation (2) can be rewritten as

$$\frac{dq}{q} = -\frac{1}{RC} dt \tag{3}$$

Integrating eq. (3) from $t = 0$ to an arbitrary later time, we find

$$\int_{q_o}^{q} \frac{dq'}{q'} = -\frac{1}{RC} \int_0^t dt'$$

or

$$\ln q' \Big|_{q_o}^{q} = \ln \left(\frac{q}{q_o} \right) = -\frac{t}{RC}$$

so that

$$q(t) = q_o e^{-t/RC} \tag{4}$$

Equations (1) and (4) imply that the charge varies with time as

$$q(t) = CV_o e^{-t/RC}$$

which was to be shown.

26-18

Referring to Fig. 26-5, we see that the current in the resistor is given by $i = dq/dt$, where $q(t)$ is the charge on the capacitor. As stated in Exercise 26-17, after the switch is thrown the charge is given by $q(t) = CV_o e^{-t/RC}$. Therefore the current in the resistor is

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}(CV_o e^{-t/RC}) = -\frac{V_o}{R} e^{-t/RC}$$

The rate at which energy is dissipated into Joule heat is given by

$$P(t) = i^2(t)R = \frac{V_o^2}{R} e^{-2t/RC}$$

The total energy dissipated in the resistor is

$$E = \int_0^\infty P(t)\,dt = \frac{V_o^2}{R}\int_0^\infty e^{-2t/RC}\,dt$$

$$= \frac{V_o^2}{R}\left[\left(-\frac{RC}{2}\right)\left(e^{-2t/RC}\Big|_0^\infty\right)\right]$$

$$= -\frac{1}{2}CV_o^2(e^{-\infty} - e^{-0}) = \frac{1}{2}CV_o^2$$

Since $q = CV_o$, we can rewrite the energy dissipated as

$$E = \frac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C}$$

which was to be shown.

26-19

In an LC circuit, the maximum current is given by

$$i_{max} = \omega_o q_o = \omega_o CV_o \tag{1}$$

Here ω_o is the resonant frequency and q_o is the maximum charge on the capacitor. We have used $q_o = CV_o$, where V_o is the original potential difference across the capacitor. Since the oscillation frequency is to be 1000 Hz, we must have $\omega_o = 2000\pi$ rad/s. With $i_{max} = 10.0$ A and $V_o = 100.0$ V, we solve eq. (1) for the required capacitance:

$$C = \frac{i_{max}}{\omega_o V_o} = \frac{(10.0)}{(2000\pi)(100.0)} = 1.59 \times 10^{-5}\ F = \underline{15.9\ \mu F}$$

The resonant frequency of an LC circuit is $\omega_o = 1/\sqrt{LC}$, so $L = (\omega_o^2 C)^{-1}$. Using eq. (1) we can write this as

$$L = \frac{1}{\omega_o^2 C} = \frac{1}{\omega_o^2} \frac{1}{(i_{max}/\omega_o V_o)} = \frac{V_o}{\omega_o i_{max}} \qquad (2)$$

With the given numerical values, we obtain

$$L = \frac{(100.0)}{(2000\pi)(10.0)} = 1.59 \times 10^{-3}\ H = \underline{1.59\ mH}$$

26-20

Referring to Fig. 26E-20, we let $i_L(t)$ denote the current (positive rightward) in the RL branch and $i_C(t)$ denote the current (positive rightward in the RC branch. Then the current (positive upward) through the battery is $i(t) = i_L(t) + i_C(t)$. The current through the RL branch is unaffected by the presence of the RC branch, so we may directly apply Eq. (26-5):

$$i_L(t) = \frac{V}{R}\left[1 - e^{-(R/L)t}\right] \qquad (1)$$

Similarly, the current in the RC branch is given by Eq. (26-13):

$$i_C(t) = \frac{V}{R} e^{-t/RC} \qquad (2)$$

Hence the current through the battery is

$$i(t) = i_L(t) + i_C(t) = \frac{V}{R}\left[1 - e^{-(R/L)t} + e^{-t/RC}\right] \qquad (3)$$

According to Eq. (26-6), the time constant of the RL branch is $\tau_L \equiv L/R$. According to Eq. (26-14), the time constant of the RC branch is $\tau_C \equiv RC$. If these have the common value $\tau_L = \tau_C = \tau$, then eq. (3) becomes

$$i(t) = \frac{V}{R}\left[1 - e^{-t/\tau} + e^{-t/\tau}\right] = \frac{V}{R}$$

for all $t > 0$. This establishes the desired result.

26-21

(a) At $t = 0$, there is a steady current downward through the inductor: $i(0) = V_o/R$ and $di/dt(0) = 0$. For $t \geq 0$, Kirchoff's voltage law for the LC circuit is

$$L\frac{di(t)}{dt} + \frac{q(t)}{C} = 0 \qquad (1)$$

Here q(t) is the charge on the lower plate of the capacitor in Fig. 26E-21. Since $i(t) = dq/dt$, eq. (1) implies

$$\frac{d^2 i(t)}{dt^2} = -\frac{i(t)}{LC} = -\omega_o^2 i(t) \qquad (2)$$

where $\omega_o \equiv 1/\sqrt{LC}$. The general solution of eq. (2) is a sinusoid:

$$i(t) = i_{max} \cos(\omega_o t + \varphi)$$

Using the initial conditions on the current, we find $i_{max} = V_o/R$ and $\varphi = 0$, so the current is

$$i(t) = \frac{V_o}{R} \cos(\omega_o t) \qquad (3)$$

(b) We are told that the charge on the lower capacitor plate is zero initially: $q(0) = 0$. The charge at time $t \geq 0$ is given by

$$q(t) = q(0) + \int_0^t \frac{dq(t')}{dt'} dt'$$

$$= 0 + \int_0^t i(t')dt'$$

$$= \frac{V_o}{R} \int_0^t \cos(\omega_o t')dt'$$

$$= \frac{V_o}{\omega_o R} \left[\sin(\omega_o t') \Big|_0^t \right] = \frac{V_o}{\omega_o R} \sin(\omega_o t) \qquad (4)$$

(c) The total energy E(t) stored in the LC circuit is the sum of the magnetic energy $\frac{1}{2}Li^2$ and the electrostatic energy $q^2/2C$. Using eqs. (3) and (4), we find

$$E(t) = \frac{Li^2(t)}{2} + \frac{q^2(t)}{2C}$$

$$= \frac{1}{2} \frac{LV_o^2}{R^2} \cos^2(\omega_o t) + \frac{V_o^2}{2C\omega_o^2 R^2} \sin^2(\omega_o t)$$

Since $\omega_o \equiv 1/\sqrt{LC}$, the energy can be rewitten as

$$E(t) = \frac{1}{2} \frac{LV_o^2}{R^2} \cos^2(\omega_o t) + \frac{1}{2} \frac{LV_o^2}{R^2} \sin^2(\omega_o t)$$

$$= \frac{1}{2} \frac{LV_o^2}{R^2}$$

26-22

We must show that the function

$$q(t) = Ae^{-\beta t/2} \cos(\omega_o t + \delta) \quad (1)$$

with

$$\omega_o \equiv \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad (2)$$

and

$$\beta \equiv R/L \quad (3)$$

satisfies the differential equation

$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \quad (4)$$

in the case of light damping ($R^2 < 4L/C$). From eq. (1), we find

$$\frac{dq}{dt} = A\left[-\frac{\beta}{2} e^{-\beta t/2} \cos(\omega_o t + \delta) - \omega_o e^{-\beta t/2} \sin(\omega_o t + \delta)\right] \quad (5)$$

and

$$\frac{d^2q}{dt^2} = A\left[\frac{\beta^2}{4} e^{-\beta t/2} \cos(\omega_o t + \delta) + 2\left(-\frac{\beta}{2}\right)(-\omega_o)\, e^{-\beta t/2} \sin(\omega_o t + \delta) - \omega_o^2 e^{-\beta t/2} \cos(\omega_o t + \delta)\right]$$

$$= Ae^{-\beta t/2}\left[\left(\frac{\beta^2}{4} - \omega_o^2\right)\cdot \cos(\omega_o t + \delta) + \beta\omega_o \sin(\omega_o t + \delta)\right] \quad (6)$$

Using eqs. (1), (5), and (6), we find that

$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$$

....

$$= Ae^{-\beta t/2}\Big[L\left(\frac{\beta^2}{4} - \omega_o^2\right)\cdot\cos(\omega_o t + \delta)$$

$$+ L\beta\omega_o \sin(\omega_o t + \delta) - \frac{R\beta}{2}\cos(\omega_o t + \delta)$$

$$- R\omega_o \sin(\omega_o t + \delta) + \frac{1}{C}\cos(\omega_o t + \delta)\Big]$$

$$= Ae^{-\beta t/2}\Big\{[\cos(\omega_o t + \delta)]\Big[L\left(\frac{\beta^2}{4} - \omega_o^2\right)$$

$$- \frac{R\beta}{2} + \frac{1}{C}\Big] + [\sin(\omega_o t + \delta)]\,[L\beta\omega_o - R\omega_o] \qquad (7)$$

Using eq. (3), we see that the coefficient of the sine term in eq. (7) is

$$L\beta\omega_o - R\omega_o \equiv L(R/L)\,\omega_o - R\omega_o = 0$$

Furthermore, eqs. (2) and (3) show that

$$L\left(\frac{\beta^2}{4} - \omega_o^2\right) - \frac{R\beta}{2} + \frac{1}{C} = L\cdot\left[\frac{R^2}{4L^2} - \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)\right] - \frac{R^2}{2L} + \frac{1}{C}$$

$$= -\frac{1}{C} + \frac{R^2}{2L} - \frac{R^2}{2L} + \frac{1}{C} = 0$$

Hence the coefficient of the cosine term in eq. (7) also vanishes. This shows that if q(t) is given by eq. (1), then eq. (4) is satisfied identically.

26-23

With $E(t) = \frac{1}{2}Li^2(t) + q^2(t)/2C$, we find

$$\frac{dE}{dt} = \frac{1}{2}L\left[2i(t)\cdot\frac{di(t)}{dt}\right] + \frac{1}{2C}\left[2q(t)\cdot\frac{dq}{dt}(t)\right] \qquad (1)$$

But $dq/dt = i$, so eq. (1) yields

$$\frac{dE(t)}{dt} = \left[L\frac{di(t)}{dt} + \frac{q(t)}{C}\right] i(t) \qquad (2)$$

However, Eq. (26-31) states that

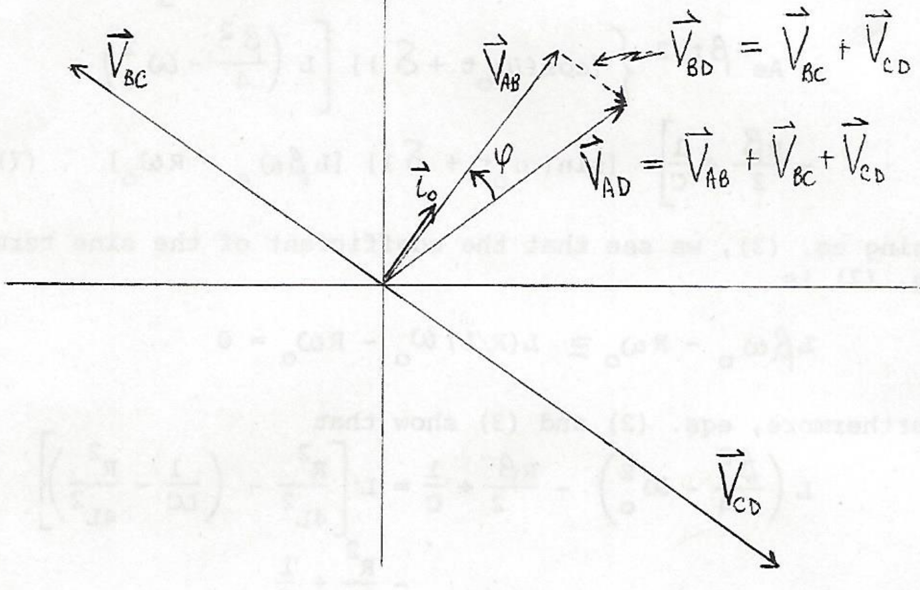
$$L\frac{di}{dt}(t) + \frac{q(t)}{C} = -Ri(t) \qquad (3)$$

Equations (2) and (3) imply that the energy stored in an undriven series LRC circuit changes at a rate

$$\frac{dE}{dt} = [-\,Ri(t)]i(t) = -i^2(t)\cdot R$$

which was to be shown.

26-24



The required phasor diagram is given above. The orientation chosen for the current phasor $\vec{i}_o$ and the voltage phasor $\vec{V}_{AB}$ across the resistor is a matter of convenience. The magnitudes of the phasors for the various circuit elements are as follows: $V_{AB} = i_o R$ = 3.0 volts, $V_{BC} = i_o X_L$ = 4.0 volts, and $V_{CD} = i_o X_C$ = 5.0 volts. As discussed in Sec. 26-7, the phasor $\vec{V}_{BC}$ across the inductor leads the current phasor by 90°, while the phasor $\vec{V}_{CD}$ trails $\vec{i}_o$ by 90°. The phasor $\vec{V}_{BD} = \vec{V}_{BC} + \vec{V}_{CD}$ has magnitude 5.0 - 4.0 = 1.0 volt. The resultant phasor $\vec{V}_{AD} = \vec{V}_{AB} + \vec{V}_{BD}$ represents the driving voltage. Its magnitude is given by

$$V_{AD} = \sqrt{V_{AB}^2 + V_{BD}^2} = \sqrt{(3.0)^2 + (1.0)^2} = \sqrt{10}$$

$$= 3.16 \text{ volts}$$

As can be seen from the phasor diagram, the current leads the driving voltage by an angle $\varphi = \tan^{-1}(V_{BD}/V_{AB}) = \tan^{-1}(1/3) = 18.4°$.

26-25

The resonant frequency of the first circuit is given by Eq. (26-58) as

$$\omega_{r1} = \frac{1}{\sqrt{L_1 C_1}} \quad (1)$$

According to the exercise statement, this is the same as the resonant frequency of the second circuit:

$$\frac{1}{\sqrt{L_2 C_2}} = \omega_{r2} = \omega_{r1} \quad (2)$$

If the six circuit elements are connected in series, the resultant circuit has inductance $L' = L_1 + L_2$, resistance $R' = R_1 + R_2$, and capacitance $C' = C_1C_2/(C_1 + C_2)$. (These results are easily established by applying Kirchoff's voltage law to the entire circuit and using the known current-voltage relationships of the indicidual circuit elements.) The resonant frequency of the new circuit is given by

$$(\omega_r')^2 = \frac{1}{L'C'} = \frac{C_1 + C_2}{C_1 C_2 (L_1 + L_2)} \quad (3)$$

Since $\omega_{r2} = \omega_{r1}$, we have $L_2C_2 = L_1C_1$, so that $C_1C_2(L_1 + L_2) = C_2^2L_2 + C_1C_2L_2 = L_2C_2(C_1 + C_2)$. Therefore, eq. (3) becomes

$$(\omega_r')^2 = \frac{C_1 + C_2}{L_2 C_2 (C_1 + C_2)} = \frac{1}{L_2 C_2} = \omega_{r2}^2$$

That is, the new resonant frequency ω_r' equals the common resonant frequency ω_{r2} of circuits 1 and 2. [NOTE: It can also be shown that if two lightly damped series LRC circuits have equal resonant frequencies ($\omega_{r1} = \omega_{r2}$) and equal natural frequencies ($\sqrt{1/L_1C_1 - 4R_1^2/L_1^2} \equiv \omega_{o1} = \omega_{o2} \equiv \sqrt{1/L_2C_2 - 4R_2^2/L_2^2}$), then not only does $\omega_r' = \omega_{r1} = \omega_{r2}$, but also the natural frequency of the six-element series circuit equals that of either of the original circuits: $\omega_o' = \omega_{o1} = \omega_{o2}$. The reader should also recognize that the mere equality of natural frequencies ($\omega_{o1} = \omega_{o2}$) does not imply that the six-element series circuit will exhibit the same natural frequency.]

26-26

We are given a driving voltage of amplitude $V_o = 150$ volts and an angular frequency $\omega = 2\pi(60\text{ Hz}) = 120\pi$ rad/s. As indicated in Fig. 26E-26, the inductor has inductance $L = 7.0$ H, and each capacitor has capacitance $C = 2.0\ \mu F$.

(i) With the contact H at point B, the bulb itself is the only element through which the source must drive current. Hence the impedance of the circuit is $Z_B = R = 120\ \Omega$, and the current has amplitude $V_o/Z_B = (150)/(120) = \underline{1.25\ A}$.

(ii) With the contact at point D, we have a series RL circuit. The inductive reactance is $X_L \equiv \omega L = (120\pi)(7.0) = 2639\ \Omega$. The impedance of the circuit is

$$Z_D = \sqrt{R^2 + X_L^2} = \sqrt{(120)^2 + (2639)^2} = 2642\ \Omega$$

Hence the current has amplitude $V_o/Z_D = (150)/(2642)$ $= \underline{5.68 \times 10^{-2}\ A}$.

(iii) With the contact at point E, we have a series RLC circuit. The inductive reactance is $X_L = 2639\ \Omega$ as before, but there is also a capacitive reactance $X_C \equiv 1/\omega C = 1326\ \Omega$. The net reactance is inductive: $X_L - X_C = 1313\ \Omega$, and the impedance is

$$Z_E = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{(120)^2 + (1313)^2} = 1318\ \Omega$$

Hence the current has amplitude $V_o/Z_E = (150)/(1318) = \underline{0.114\ A}$.

(iv) With the contact at point F, we have a series RLC circuit. The capacitance of two $2.0\text{-}\mu F$ capacitors in series is $C_F = 1.0\ \mu F$. Hence the capacitive reactance $X_C = 1/\omega C_F = 1/[(120\pi)(1.0 \times 10^{-6})] = 2653\ \Omega$. The net reactance is capacitive: $X_L - X_C = -14\ \Omega$, and the impedance is

$$Z_F = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{(120)^2 + (-14)^2} = 121\ \Omega$$

Hence the current has amplitude $V_o/Z_F = (150)/(121) = \underline{1.24\ A}$.

(v) With the contact at point G, the capacitance is $C_G = 1/3(2.0\ \mu F)$ and the capacitive reactance is $X_C = 1/\omega C_G = 3/[(120\pi)(2.0 \times 10^{-6})] = 3979\ \Omega$. The net reactance is capacitive: $X_L - X_C = -1340\ \Omega$, and the impedance is

$$Z_G = \sqrt{R^2 + (X_L - X_C)^2}$$

....

$$= \sqrt{(120)^2 + (-1340)^2} = 1345\ \Omega$$

Hence the current has amplitude $V_o/Z_G = (150)/(1345) = \underline{0.112\ A}$.

(b) With the contact H at point F, we have $R = 120\ \Omega$, $X_L = 2639\ \Omega$, $X_C = 2652\ \Omega$, and a peak current $i = 1.24$ A. Therefore the voltage between B and D has amplitude $|\vec{V}_{BD}| = iX_L = (1.24)(2639)$ $= 3.27 \times 10^3$ V. The voltage between D and F has amplitude $|\vec{V}_{DF}|$ $= iX_C = (1.24)(2653) = 3.29 \times 10^3$ V. The sum of these amplitudes is $|\vec{V}_{BD}| + |\vec{V}_{DF}|$, which is more than 6500 volts! However, these voltages are 180° out of phase, so that the magnitude of the voltage phasor $\vec{V}_{BF}$ is $V_{BF} = \left| |\vec{V}_{BD}| - |\vec{V}_{DF}| \right|$. This is less than 20 volts.

<u>26-27</u>

The stored energy E is equal to the energy in the magnetic field of the inductor at the instant of maximum current:

$$E = \frac{1}{2} Li_o^2 \qquad (1)$$

The average rate of energy dissipation in the resistance is P_{av} $= i_{rms}^2 R$. Since $i_{rms}^2 = \frac{1}{2} i_o^2$ for a sinusoidally varying current, the average power in the resistor is $P_{av} = \frac{1}{2} i_o^2 R$. Consequently the energy dissipated in each cycle of oscillation is

$$P_{av} T = \frac{1}{2}\, i_o^2 RT \qquad (2)$$

where T is the period. According to the definition given in Sec. 26-8, the Q factor is

$$Q = \frac{2\pi E}{P_{av} T} = \frac{2\pi (Li_o^2/2)}{(i_o^2 RT/2)} = \frac{2\pi}{T}\frac{L}{R} = \frac{L\omega}{R} \qquad (3)$$

since $\omega T = 2\pi$. This establishes Eq. (26-75b), which was written under the assumption that the system is driven at or near resonance: $\omega \simeq \omega_r$.

<u>26-28</u>

(a) Since both the current and the voltage vary sinusoidally, Eq. (26-63) can be rewritten in terms of rms quantities: i_{rms} $= V_{rms}/Z$. Solving for the impedance, we have

$$Z = V_{rms}/i_{rms} = (110)/(0.60) = 183.33 = \underline{183\ \Omega}$$

(b) According to Eq. (26-80b), the average power $\langle P \rangle$ $= V_{rms} i_{rms} \cos\varphi$. Hence the power factor is

$$\cos\varphi = \frac{\langle P\rangle}{V_{rms} i_{rms}} = \frac{36}{(110)(0.60)} = \underline{0.545}$$

(c) Solving Eq. (26-72b) for the resistance, we have

$$R = Z\cos\varphi = \frac{V_{rms}}{i_{rms}}\left(\frac{\langle P\rangle}{V_{rms} i_{rms}}\right) = \frac{\langle P\rangle}{i_{rms}^2}$$

$$= \frac{36}{(0.60)^2} = \underline{100\ \Omega}$$

(d) Since there is no capacitive reactance in the circuit, the net reactance $X = X_L$, and Eq. (26-72a) becomes $Z = (R^2 + X_L^2)^{1/2}$. Solving this for X_L, we obtain

$$X_L = (Z^2 - R^2)^{1/2} = [(183.33)^2 - (100)^2]^{1/2}$$

$$= 153.7\ \underline{\Omega}$$

Since $X_L = \omega L$, we have $L = X_L/\omega$. With $\omega = 2\pi$ (60 Hz) $= 120\pi$ rad/s, we obtain

$$L = \frac{X_L}{\omega} = \frac{153.7}{120\pi} = \underline{0.408\ \text{H}}$$

26-29

(a) The resistance of the 100-W, 120-V light bulb is given by $R_A = (120)^2/100 = 144\ \Omega$. The rms current it would draw at full power is (100 W)/(120 V) = 0.83 A. The impedance of the circuit (with switch S open) is given by $Z = \sqrt{R_A^2 + X_L^2}$, where $X_L = \omega L_p = (2\pi)(60)(10) = 3770\ \Omega$. Since $X_L \gg R_A$, we have

$$Z \simeq X_L\left[1 + \frac{1}{2}\left(\frac{R_A}{X_L}\right)^2\right] = 3770\left[1 + \frac{1}{2}\left(\frac{144}{3770}\right)^2\right]$$

$$= 3773\ \Omega \simeq 3.77 \times 10^3$$

Therefore when the switch S is open, the rms current in the primary circuit is given by $(120)/(3.77 \times 10^3) \simeq 0.032$ A, which is much less than the 0.83 A required for full bulb brightness. The power dissipated in bulb A is $(0.032)^2(144) = 0.15$ W, which is less than 0.2% of the bulb's nominal power!

(b) At full power, bulb B draws an rms current of (60 W)/(12 V) = 5.0 A. Since bulb B glows visibly, there is evidently an induced emf in the secondary coil which drives a secondary current that is a significant fraction of 5.0 A. By Lenz' law, the induced current flows through the secondary coil in a direction that at each instant opposes the flux change that induced the emf. Referring to Fig. 26E-29, the induced current in the secondary not only reduces the flux change in the secondary coil but also reduces the (initiating) flux change in the primary. As a result, the back emf in the primary is reduced. (With switch S open, the amplitude of the back emf in the primary was essentially equal to the 120-V amplitude of the driving voltage.) With the reduction in back emf, the power supply is able to drive a significant current through bulb A. [NOTE: Neglecting any dissipation of energy within the transformer coils and core, the time-averaged product $\langle i_1(t)V_s(t)\rangle$, where i_1 is the primary current and $V_s(t)$ is the source voltage, equals the sum of $\langle i_1^2(t)\rangle R_A$, the power dissipated in bulb A, and $\langle i_2^2(t)\rangle R_B$, the power dissipated in bulb B.]

26-30

We let ℓ denote the length of the central line of the toroid. Assuming that coil 1 contains N_1 turns and carries current i_1, Ampere's law can be used to show that the toroidal field contribution of coil 1 has magnitude

$$\mathcal{B}_1 = \frac{K_m \mu_o N_1 i}{\ell} \tag{1}$$

along the central line. Here K_m is the relative permeability of the iron. Neglecting the variation of $\mathcal{B}_1$ over the cross section of the core, and letting $\underline{a}$ denote the cross-sectional area, we use eq. (1) to find the flux linking each loop of coil 1:

$$\Phi_1 = \mathcal{B}_1 a = \frac{K_m \mu_o N_1 i_1 a}{\ell} \tag{2}$$

Using eq. (2) and the definition of self-inductance (see Sec. 25-5), we find

$$L_1 = \frac{N_1 \Phi_1}{i_1} = \frac{K_m \mu_o N_1^2 a}{\ell} \tag{3}$$

Similarly, we must have

$$L_2 = \frac{N_2 \Phi_2}{i_2} = \frac{K_m \mu_o N_2^2 a}{\ell} \tag{4}$$

The magnitude of the mutual inductance is given by

$$|M_{12}| = \frac{N_2 \Phi_{12}}{i_1} \tag{5}$$

as discussed in Sec. 25-5. The flux (due to coil 1) that links each loop of coil 2 is equal to Φ_1 if no magnetic field lines leave the torus. Then eqs. (2) and (5) imply

$$\left|M_{12}\right| = \frac{N_2}{i_1}\left(\frac{K_m \mu_o N_1 i_1 a}{\ell}\right) = \frac{K_m \mu_o N_1 N_2 a}{\ell} \quad (6)$$

Comparing eqs. (3), (4), and (6), we see that

$$\left|M_{12}\right|^2 = L_1 L_2 \quad (7)$$

Supposing that the sign conventions for voltages are chosen in such a way that $M_{12} > 0$, eq. (7) yields $\underline{M_{12} = \sqrt{L_1 L_2}}$, as desired.

26-31

(a) For $0 \leq t \leq T$, the Kirchoff's voltage law for the circuit has the form

$$iR + L\frac{di}{dt} = \left|V_A\right|$$

Since $i(0) = 0$, the discussion of Sec. 26-2, including Eq. (26-5), is applicable:

$$i(t) = \frac{\left|V_A\right|}{R}\left(1 - e^{-t/\tau_L}\right) \quad (1)$$

where $\tau_L \equiv L/R$.

(b) Because an inductor is present in the circuit, the current cannot vary discontinuously. Hence the current given in eq. (1), as evaluated at $t = T$, serves as an "initial condition" on the circuit's behavior for $t \geq T$. We have

$$i(T) = \frac{\left|V_A\right|}{R}\left(1 - e^{-T/\tau_L}\right) \quad (2)$$

Once the switch is thrown to position B, Kirchoff's voltage law reads

$$iR + L\frac{di}{dt} = -\left|V_B\right| \quad (3)$$

The solution of eq. (3) is the sum of the "particular solution" $i_p(t) = -\left|V_B\right|/R$ and the general solution to the homogeneous equation $i_h R + L di_h/dt = 0$. We find $i_h(t) = Ae^{-t/\tau_L}$, where A is an arbitrary constant. The solution of eq. (3) is therefore

$$i(t) = -\frac{\left|V_B\right|}{R} + Ae^{-t/\tau_L} \quad (4)$$

The constant A is determined by the requirement that eq. (4) be consistent with eq. (2) -- that is, by imposing the "initial condition" at t = T:

$$\frac{|V_A|}{R}(1 - e^{-T/\tau_L}) = -\frac{|V_B|}{R} + Ae^{-T/\tau_L}$$

Solving for A, we obtain

$$A = \frac{(|V_A| + |V_B|)}{R} e^{T/\tau_L} - \frac{|V_A|}{R} \qquad (5)$$

Substituting eq. (5) into eq. (4), we find

$$i(t) = -\frac{|V_B|}{R} + e^{-t/\tau_L}\left[\frac{(|V_A| + |V_B|)}{R} e^{T/\tau_L} - \frac{|V_A|}{R}\right]$$

$$= \frac{|V_A|}{R}(1 - e^{-T/\tau_L})e^{-(t-T)/\tau_L}$$

$$- \frac{|V_B|}{R}\left[1 - e^{-(t-T)/\tau_L}\right] \qquad (6)$$

(c) The switch can be opened safely only when the current given by eq. (6) is instantaneously zero. This requires

$$\frac{|V_A|}{R}(1 - e^{-T/\tau_L})e^{-(t'-T)/\tau_L} = \frac{|V_B|}{R}\left[1 - e^{-(t'-T)/\tau_L}\right]$$

or

$$e^{-(t'-T)/\tau_L} = \frac{|V_B|}{|V_B| + |V_A|\cdot(1 - e^{-T/\tau_L})}$$

or

$$\frac{t' - T}{\tau_L} = \ln\left[1 + \frac{|V_A|}{|V_B|}\cdot(1 - e^{-T/\tau_L})\right]$$

or

$$t' = T + \tau_L \ln\left[1 + \frac{|V_A|}{|V_B|}\cdot(1 - e^{-T/\tau_L})\right] \qquad (7)$$

<u>26-32</u>

(a) The current i(t) in the circuit is zero for $t < 0$. For $0 \leq t < T$, it is governed by the differential equation

$$L\frac{di}{dt} + Ri = V_o \qquad (1)$$

For $t \geq T$, it is governed by the differential equation

$$L \frac{di}{dt} + Ri = 0 \qquad (2)$$

[Equations (1) and (2) simply state Kirchoff's voltage rule for the circuit.] Since the current vanishes for $t<0$, the present problem is identical to the one solved in Sec. 26-2, for $0 \leq t < T$. That is, Eq. (26-5) provides the desired solution for eq. (1) above, subject to $i(0) = 0$:

$$i(t) = \frac{V_o}{R}(1 - e^{-t/\tau_L}) \qquad (3)$$

for $0 \leq t < T$, where $\tau_L \equiv L/R$. The solution of eq. (2) is the sum of the particular solution $i_p(t) = 0$ and the general solution to the homogeneous equation: $L di_h/dt + Ri_h = 0$. We find $i_h(t) = Ae^{-t/\tau_L}$, where A is an arbitrary constant. It is convenient to rewrite this as $i_h(t) = A'e^{-(t-T)/\tau_L}$, where $A'e^{T/\tau_L} \equiv A$. Then the general solution of eq. (3) is

$$i(t) = i_p(t) + i_h(t) = 0 + A'e^{-(t-T)/\tau_L} \qquad (4)$$

Equation (4) describes the current for $t>T$. Since there is an inductor present, the current cannot vary discontinuously. Therefore we may equate expressions for $i(T)$ obtained from eqs. (3) and (4):

$$\frac{V_o}{R}(1 - e^{-T/\tau_L}) = A'$$

Hence for $t>T$, the current in the circuit is given by

$$i(t) = \frac{V_o}{R}(1 - e^{-T/\tau_L})e^{-(t-T)/\tau_L} \qquad (5)$$

(b) The voltage $V_R(t)$ is given by $V_R(t) = i(t)R$, so eqs. (3) and (5) imply

$$V_R(t) = V_o(1 - e^{-t/\tau_L}) \qquad (6)$$

for $0<t<T$, and

$$V_R(t) = V_o(1 - e^{-T/\tau_L})e^{-(t-T)/\tau_L} \qquad (7)$$

for $t>T$. The voltage $V_L(t)$ is given by $V_L(t) = L di/dt$, so eqs. (3) and (5) imply

$$V_L(t) = L\frac{d}{dt}\left[\frac{V_o}{R}(1 - e^{-t/\tau_L})\right] \;\ldots\ldots\ldots$$

$$= \frac{LV_o}{R}(-)\left(-\frac{1}{\tau_L}\right)\cdot e^{-t/\tau_L} = V_o e^{-t/\tau_L} \quad (8)$$

for $0 < t < T$. Similarly, for $t > T$, we obtain

$$V_L(t) = L\frac{d}{dt}\left[\frac{V_o}{R}(1 - e^{-T/\tau_L})e^{-(t-T)/\tau_L}\right]$$

$$= \frac{LV_o}{R}(1 - e^{-T/\tau_L})e^{T/\tau_L}\left(-\frac{1}{\tau_L}\right)e^{-t/\tau_L}$$

$$= -V_o(1 - e^{-T/\tau_L})e^{-(t-T)/\tau_L} \quad (9)$$

(c) Using eqs. (6) and (8), we find that for $0 < t < T$,

$$V_R(t) + V_L(t) = V_o(1 - e^{-t/\tau_L}) + V_o e^{-t/\tau_L} = V_o$$

as desired. For $t > T$, eqs. (7) and (9) show that $V_R(t) + V_L(t) = 0$, as desired.

26-33

(a) Referring to Fig. 26E-33, we let i_C be the (positive downward) current in the capacitor. During times when the glow tube N is nonconducting, Kirchoff's voltage rule takes the form

$$V_b = i_C R + \frac{q_C(t)}{C} \quad (1)$$

Taking $t = 0$ at an instant when the capacitor has just been discharged, we have $q_C(0) = 0$. Then eq. (1) implies that $i_C(0) = V_b/R$. As the capacitor accumulates charge, the current will gradually diminish. To solve for the precise variation of the current, we differentiate eq. (1), obtaining

$$R\frac{di_C}{dt} + \frac{i_C}{C} = 0 \quad (2)$$

or

$$\frac{di_C}{i_C} = -\frac{dt}{RC} \quad (3)$$

Integrating eq. (3), we find

$$\ln[i_C(t)/i_C(0)] = -t/RC \quad (4)$$

With $i_C(0) = V_b/R$, eq. (4) implies

$$i_C(t) = \frac{V_b}{R} e^{-t/RC} \qquad (5)$$

Eq. (1) then implies

$$V_C(t) = \frac{q_C(t)}{C} = V_b - i_C(t)\cdot R$$

$$= V_b(1 - e^{-t/RC}) \qquad (6)$$

as was to be shown. Equation (6) simply describes the charging of a capacitor by a steady voltage source. [It remains valid until $V_C(t)$ reaches the value V_f.]

(b) The power series expansion for the exponential function yields

$$e^{-t/RC} = \sum_{n=0}^{\infty} \frac{(-t/RC)^n}{n!} = 1 - \frac{t}{RC} + \frac{1}{2}\left(\frac{t}{RC}\right)^2 - \frac{1}{6}\left(\frac{t}{RC}\right)^3 + \dots$$

Therefore the voltage on the capacitor can be written as

$$V_C(t) = V_b\left\{1 - \left[1 - \frac{t}{RC} + \frac{1}{2}\left(\frac{t}{RC}\right)^2 - \dots\right]\right\}$$

For $t \ll RC$, this is given to good accuracy by

$$V_C(t) \simeq \frac{V_b t}{RC} \qquad (7)$$

[The fractional error is approximately $(t/2RC)$.]

(c) Assuming that $V_f \ll V_b$, eq. (7) is valid for $0 < V_C(t) \leq V_f$. Then the capacitor voltage increases linearly with time until $V_C(t) = V_f$, at which time the glow lamp N effectively "shorts out" the capacitor. (Since its resistance $R_N \ll R$, the drop in the voltage curve is almost vertical, as indicated in Fig. 26E-33a.) After the capacitor has been discharged, the glow lamp "reopens" and voltage source once again begins to charge up the capacitor. The period T of the sawtooth is the time required for V_C to rise from 0 to V_f. Using eq. (7), we have $V_f = V_b T/RC$, or

$$T = (V_f/V_b)RC \qquad (8)$$

which was to be shown.

26-34

The charge on the capacitor in a lightly damped series LRC circuit that is undriven after t = 0 is given by Eq. (26-42a):

$$q(t) = Ae^{-(\beta/2)t} \cos(\omega_o t + \delta) \qquad (1)$$

The corresponding current is given by Eq. (26-45):

$$i(t) = -\,Ae^{-(\beta/2)t}\left[\frac{1}{2}\beta\cos(\omega_o t + \delta) + \omega_o \sin(\omega_o t + \delta)\right] \tag{2}$$

The natural frequency ω_o is given by Eq. (26-43):

$$\omega_o \equiv \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

and the quantity $\beta \equiv R/L$. The quantities A and δ which appear in eqs. (1) and (2) are not determined by Kirchoff's voltage rule for the circuit. Rather they are determined by the initial conditions in the circuit. In the present case, we are given $q(0) = q_o$ and $i(0) = 0$, so that eqs. (1) and (2) become

$$q_o = A\cos\delta \tag{3}$$

and

$$0 = -\,A\left[\frac{1}{2}\beta\cos\delta + \omega_o \sin\delta\right] \tag{4}$$

Equation (4) is equivalent to

$$\tan\delta = -\frac{\beta}{2\omega_o} = -\frac{R}{2L\omega_o} \tag{5}$$

Since $\cos^2\delta = (1 + \tan^2\delta)^{-1}$, we find

$$\cos^2\delta = \left(1 + \frac{R^2}{4L^2\omega_o^2}\right)^{-1}$$

$$= \frac{\omega_o^2}{\omega_o^2 + \frac{R^2}{4L^2}} = \frac{\omega_o^2}{\frac{1}{LC} - \frac{R^2}{4L^2} + \frac{R^2}{4L^2}} = \omega_o^2 LC$$

so that

$$\cos\delta = \pm\,\omega_o\sqrt{LC} \tag{6}$$

Adopting the (common) convention that A must be positive, the appropriate sign in eq. (6) depends on q_o. Referring to eq. (3), we find that

$$\cos\delta = \frac{q_o}{|q_o|}\,\omega_o\sqrt{LC} \tag{7}$$

and

$$A = \frac{q_o}{\cos\delta} = \frac{|q_o|}{\omega_o\sqrt{LC}} \tag{8}$$

Eqs. (5) and (7) imply that

$$\sin\delta \equiv \tan\delta\,\cos\delta = -\frac{R}{2L\omega_o}\left(\frac{q_o}{|q_o|}\,\omega_o\sqrt{LC}\right)$$

$$= -\frac{Rq_o}{2|q_o|}\sqrt{\frac{C}{L}} \tag{9}$$

Now we can evaluate eq. (2) for the current:

$$i(t) = -\frac{|q_o|}{\omega_o\sqrt{LC}}\,e^{-(\beta/2)t}\Big[\frac{1}{2}\beta\cos(\omega_o t+\delta)$$

$$+\,\omega_o\sin(\omega_o t+\delta)\Big]$$

$$= -\frac{|q_o|}{\omega_o\sqrt{LC}}\,e^{-(\beta/2)t}\Big\{\frac{1}{2}\beta[\cos(\omega_o t)\cos\delta$$

$$-\sin(\omega_o t)\sin\delta] + \omega_o[\sin(\omega_o t)\cos\delta$$

$$+\cos(\omega_o t)\sin\delta]\Big\}$$

$$= -\frac{|q_o|}{\omega_o\sqrt{LC}}\,e^{-(\beta/2)t}\Big[\Big(\frac{1}{2}\beta\cos\delta$$

$$+\,\omega_o\sin\delta\Big)\cos(\omega_o t) + \Big(-\frac{1}{2}\beta\sin\delta$$

$$+\,\omega_o\cos\delta\Big)\sin(\omega_o t)\Big] \tag{10}$$

Using eqs. (4), (7), and (8), we obtain

$$i(t) = -\frac{|q_o|}{\omega_o\sqrt{LC}}\,e^{-(\beta/2)t}\Big\{(0)\cdot\cos(\omega_o t)$$

$$+\Big[\Big(-\frac{1}{2}\beta\Big)\Big(-\frac{Rq_o}{2|q_o|}\sqrt{\frac{C}{L}}\Big) \quad \ldots\ldots\ldots$$

$$+ \omega_o \frac{q_o}{|q_o|} \omega_o \sqrt{LC} \Big] \cdot \sin(\omega_o t) \Big\}$$

$$= - \frac{|q_o|}{\omega_o \sqrt{LC}} e^{-(R/2L)t} \frac{q_o}{|q_o|} \Big(\frac{R^2}{4L} \sqrt{LC}$$

$$+ \omega_o^2 \sqrt{LC} \Big) \sin(\omega_o t)$$

Since $\omega_o^2 + R^2/4L^2 = 1/LC$, this becomes

$$i(t) = - \frac{q_o}{\omega_o \sqrt{LC}} e^{-(R/2L)t} \cdot \frac{\sqrt{LC}}{LC} \sin(\omega_o t)$$

$$= - \frac{q_o}{\omega_o LC} e^{-(R/2L)t} \sin(\omega_o t)$$

which was to be shown.

26-35

(a) (i) With a finite driving voltage, the current in an inductor must vary continuously. Since i_L is zero just before the switch is closed, it must also be zero just after the switch is closed: $i_L(0+) = \underline{0}$.

(ii) The voltage across the resistor at t = 0 is just the initial capacitor voltage: $V_R(0+) = V_C(0+) = q_o/C$, so the resistor current must be $i_R(0+) = V_R(0+)/R = \underline{q_o/RC}$.

(iii) In Fig. 26E-35, the positive sense for the current i_C has current flowing through the circuit away from the left plate of the capacitor. Hence Kirchoff's current rule requires that $i_C(t) = i_R(t) + i_L(t)$. Therefore $i_C(0+) = q_o/RC + 0 = \underline{q_o/RC}$.

(b) Since there is resistive damping in the circuit, we must expect that the energy of the circuit, which is given by

$$E(t) = \frac{L}{2} i_L^2(t) + \frac{1}{2C} q^2(t)$$

will approach zero as $t \to \infty$. Therefore $i_L \to 0$ and $q_C \to 0$. Since $q_C \to 0$, the resistor current $i_R(t) = q_C(t)/RC$ also approaches zero. Hence $i_C(t) = i_R(t) + i_L(t)$ must also approach zero.

(c) Applying Kirchoff's voltage rule, we obtain

$$L \frac{di_L}{dt}(t) = Ri_R(t) = \frac{q(t)}{C} \quad (1)$$

The current rule yields

$$i_C(t) = i_R(t) + i_L(t) \quad (2)$$

With the direction indicated in Fig. 26E-35 for the capacitor current, we have

$$i_C(t) = -\frac{dq}{dt}(t) \quad (3)$$

Differentiating eqs. (2) and (3), we find

$$\frac{di_C}{dt} = -\frac{d^2q}{dt^2} = \frac{di_R}{dt} + \frac{di_L}{dt} \quad (4)$$

Equations (1) and (4) imply that

$$-\frac{d^2q}{dt^2} = \frac{d}{dt}\left(\frac{q}{RC}\right) + \frac{q(t)}{LC}$$

or

$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{q}{LC} = 0 \quad (5)$$

as desired.

(d) We must substitute the proposed general solution,

$$q = Ae^{-(\beta_p/2)t} \cos(\omega_p t) + Be^{-(\beta_p/2)t} \sin(\omega_p t) \quad (6)$$

into the differential equation (5) to determine whether a mathematical identity can result. We find

$$\frac{dq}{dt} = A\left[-\left(\frac{\beta_p}{2}\right)e^{-(\beta_p/2)t} \cos(\omega_p t) - \omega_p e^{-(\beta_p/2)t} \times \sin(\omega_p t)\right] + B\left[-\left(\frac{\beta_p}{2}\right)e^{-(\beta_p/2)t} \sin(\omega_p t) + \omega_p e^{-(\beta_p/2)t} \cos(\omega_p t)\right] \quad (7)$$

The second derivative of eq. (6) is

$$\frac{d^2q}{dt^2} = A\left[\left(\frac{\beta_p^2}{4}\right)e^{-(\beta_p/2)t}\cos(\omega_p t) + \beta_p\omega_p e^{-(\beta_p/2)t}\right.$$

$$\left. \times \sin(\omega_p t) - \omega_p^2 e^{-(\beta_p/2)t}\cos(\omega_p t)\right]$$

$$+ B\left[\left(\frac{\beta_p^2}{4}\right)e^{-(\beta_p/2)t}\sin(\omega_p t) - \beta_p\omega_p e^{-(\beta_p/2)t}\right.$$

$$\left. \times \cos(\omega_p t) - \omega_p^2 e^{-(\beta_p/2)t}\sin(\omega_p t)\right]$$

The proposed equality is

$$0 = \frac{q}{LC} + \frac{1}{RC}\frac{dq}{dt} + \frac{d^2q}{dt^2}$$

$$= Ae^{-(\beta_p/2)t}\left[\left(\frac{1}{LC}\right)\cdot\cos(\omega_p t) - \left(\frac{\beta_p}{2RC}\right)\cdot\cos(\omega_p t)\right.$$

$$- \left(\frac{\omega_p}{RC}\right)\cdot\sin(\omega_p t) + \frac{\beta_p^2}{4}\cos(\omega_p t) + \beta_p\omega_p\sin(\omega_p t)$$

$$\left. - \omega_p^2\cos(\omega_p t)\right] + Be^{-(\beta_p/2)t}\left[\left(\frac{1}{LC}\right)\cdot\sin(\omega_p t)\right.$$

$$- \left(\frac{\beta_p}{2RC}\right)\cdot\sin(\omega_p t) + \left(\frac{\omega_p}{RC}\right)\cdot\cos(\omega_p t)$$

$$\left. + \left(\frac{\beta_p^2}{4}\right)\sin(\omega_p t) - \beta_p\omega_p\cos(\omega_p t) - \omega_p^2\sin(\omega_p t)\right]$$

Canceling the common factor of $e^{-(\beta_p/2)t}$ and rearranging terms, we rewrite the proposed equality as

$$0 = \cos(\omega_p t)\cdot\left[A\left(\frac{1}{LC} - \frac{\beta_p}{2RC} + \frac{\beta_p^2}{4} - \omega_p^2\right)\right.$$

$$\left. + B\left(\frac{\omega_p}{RC} - \beta_p\omega_p\right)\right] + \sin(\omega_p t)\cdot\left[A\left(\beta_p\omega_p - \frac{\omega_p}{RC}\right)\right.$$

$$\left. + B\left(\frac{1}{LC} - \frac{\beta_p}{2RC} + \frac{\beta_p^2}{4} - \omega_p^2\right)\right] \tag{8}$$

Equation (8) will be satisfied for all A and B provided that

$$\beta_p = \frac{1}{RC}, \tag{9}$$

and

$$\omega_p^2 = \frac{1}{LC} - \frac{\beta_p}{2RC} + \frac{\beta_p^2}{4} \qquad (10)$$

Using eq. (9) in eq. (10, we find

$$\underline{\omega_p} = \left(\frac{1}{LC} - \frac{1}{2R^2C^2} + \frac{1}{4R^2C^2}\right)^{1/2}$$

$$= \left(\frac{1}{LC} - \frac{1}{4R^2C^2}\right)^{1/2} \qquad (11)$$

Hence eq. (6) gives a solution provided that the radicand in eq. (11) is positive. This is the case if and only if $R^2 > 4L/C$. In order to establish that eq. (6) is the general solution of eq. (5) when $R^2 > 4L/C$, we can invoke a property of any linear and homogeneous second-order differential equation --- namely, that its general solution is a linear combination of exactly two independent functions. Alternatively, we can present a more intuitive and more physical argument. Referring to Fig. 26E-35, we recognize that the most general initial situation in the circuit has an arbitrary initial charge q(0-) on the capacitor and an arbitrary initial current $i_R(0-) = -i_L(0-)$ in the RL portion of the circuit. Evidently the coefficients A and B in eq. (6) can be chosen to accommodate any initial conditions q(0-) and $i_R(0-)$.

(e) As shown in part (a), the initial conditions stated in the exercise imply $q(0+) = q_o$ and $dq/dt\big|_{0+} = -q_o/RC$. From eq. (6), we find

$$\underline{q_o = A} \qquad (12)$$

From eq. (7), we find

$$-\frac{q_o}{RC} = -\frac{\beta_p A}{2} + \omega_p B$$

Solving for B with the help of eqs. (9) and (12) we obtain

$$\underline{B} = \frac{1}{\omega_p}\left(\frac{\beta_p A}{2} - \frac{q_o}{RC}\right)$$

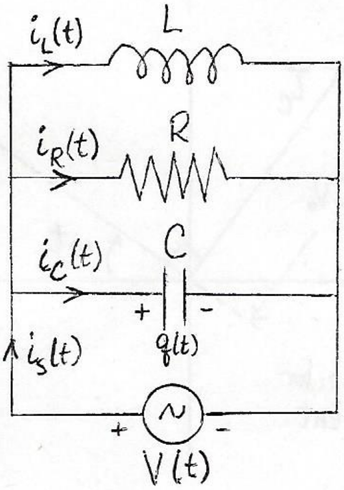
$$= \frac{1}{\omega_p}\left(\frac{q_o}{2RC} - \frac{q_o}{RC}\right) = \underline{-\frac{q_o}{2RC\,\omega_p}} \qquad (13)$$

Using eqs. (12) and (13) in eq. (6), we have the desired solution:

$$\underline{q(t) = q_o e^{-\beta_p t/2}\left[\cos(\omega_p t) - \left(\frac{1}{2RC\,\omega_p}\right)\cdot\sin(\omega_p t)\right]} \qquad (14)$$

(f) In a series LRC circuit, all of the current flows through the resistor. In order for oscillations to occur, the resistance must be so small that only a small fraction of the initial energy is dissipated during a time comparable to $2\pi\sqrt{LC}$, the natural frequency of an undamped LC circuit. In a parallel LRC circuit, the current from the capacitor is divided between the resistor and the inductor. If the resistance is too small, the resistor effectively "short-circuits" the inductor and prevents oscillations from occurring. (When a resistor has a given voltage drop -- in this case, q/C -- the dissipation rate is inversely proportional to R. As in the series circuit, the light-damping criterion ensures that there is relatively little energy dissipation during a time comparable to $2\pi\sqrt{LC}$.)

26-36



We examine the driven parallel LRC circuit of Fig. 26-22, using the sign conventions indicated at the right. All three components have the same sinusoidal voltage across them. The current i_S from the source is the sum of the currents i_L, i_R, and i_C. Hence the phasor $\vec{i}_S = \vec{i}_L + \vec{i}_R + \vec{i}_C$. With the given driving voltage $V(t) = V_o \sin(\omega t)$, the current in the resistor is

$$i_R(t) = \frac{V_o}{R}\sin(\omega t) \tag{1}$$

which is in phase with V(t). The current in the inductor satisfies $L di_L/dt = V_o \sin(\omega t)$ and has zero average value $\langle i_L(t)\rangle = 0$. This implies that

$$i_L(t) = -\frac{V_o}{\omega L}\cos(\omega t) = \frac{V_o}{\omega L}\sin(\omega t - 90°) \tag{2}$$

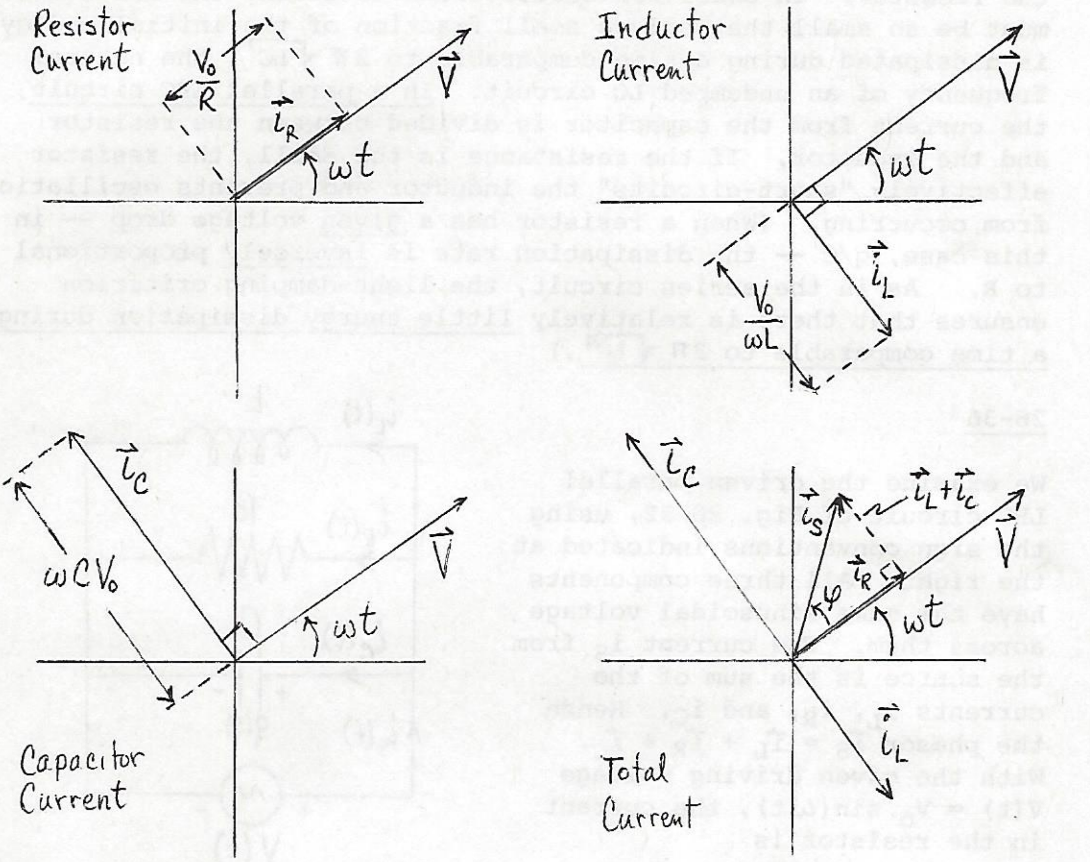
That is, the inductor current lags the voltage by 90°.

The capacitor current is given by

$$i_C(t) = \frac{dq(t)}{dt} = \frac{d}{dt}[CV(t)] = CV_o\frac{d}{dt}[\sin(\omega t)]$$

$$= \omega CV_o \cos(\omega t) = \omega CV_o \sin(\omega t + 90°) \tag{3}$$

That is, the capacitor current leads the voltage by 90°. The current-voltage relationships of eqs. (1) - (3) are exhibited in the phasor diagrams on the next page. The fourth diagram shows the relationship between the total current phasor $\vec{i}_o$ and the voltage phasor $\vec{V}$.



The magnitude of $\vec{i}_S$ is

$$\left|\vec{i}_S\right| = \sqrt{\left|\vec{i}_R\right|^2 + \left(\left|\vec{i}_C\right| - \left|\vec{i}_L\right|\right)^2}$$

$$= V_o\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

The impedance of the parallel LRC circuit is

$$Z_p = \frac{V_o}{\left|\vec{i}_S\right|} = \left[\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right]^{-1/2}$$

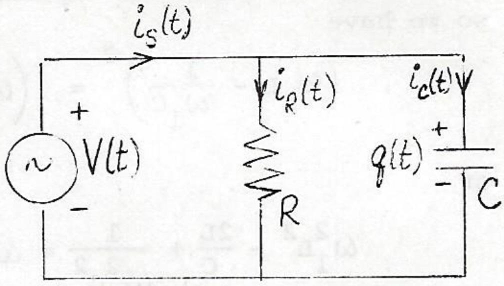
Referring to the phasor diagram, we find that the current leads the voltage by the phase angle

$$= \tan^{-1}\left(\frac{\left|\vec{i}_C\right| - \left|\vec{i}_L\right|}{\left|\vec{i}_R\right|}\right) = \tan^{-1}\left[\frac{(\omega C V_o) - (V_o/\omega L)}{(V_o/R)}\right]$$

$$= \tan^{-1}\left\{R[\omega C - (1/\omega L)]\right\}$$

[Obviously, if $\omega C < 1/\omega L$, then $\varphi < 0$ and $i_S(t)$ lags $V(t)$.]

26-37



The circuit diagram at the right specifies the notation. The total current $i_S(t)$ is the sum of the resistor current $i_R(t)$ and the capacitor current $i_C(t)$. With $V(t) = V_o \sin(\omega t)$, the resistor current must be

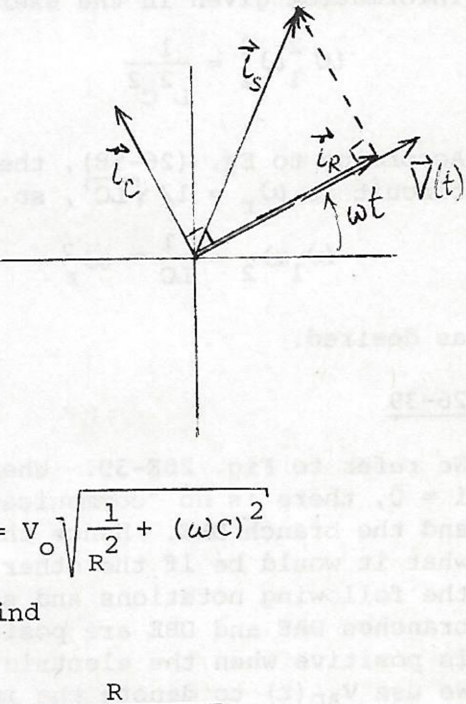
$$i_R(t) = \frac{V_o}{R}\sin(\omega t)$$

and the capacitor current must be

$$i_C(t) = \frac{dq_C}{dt}(t)$$

$$= \frac{d}{dt}[CV(t)] = CV_o\frac{d}{dt}[\sin(\omega t)] = \omega CV_o \cos(\omega t)$$

$$= \omega CV_o \sin(\omega t + 90°)$$



Hence the resistor current is in phase with the driving voltage, while the capacitor current leads the voltage by 90°. These relationships are exhibited in the phasor diagram at the right, whice also shows the total current phasor $\vec{i}_S$. From the diagram, we find

$$\left|\vec{i}_S\right| = \sqrt{\left|\vec{i}_R\right|^2 + \left|\vec{i}_C\right|^2}$$

$$= \sqrt{\left(\frac{V_o}{R}\right)^2 + (\omega CV_o)^2} = V_o\sqrt{\frac{1}{R^2} + (\omega C)^2}$$

By definition, $Z \equiv V_o/|\vec{i}_S|$, so we find

$$Z = \frac{V_o}{\left[V_o\sqrt{(1/R^2) + \omega^2C^2}\right]} = \frac{R}{\sqrt{1 + \omega^2R^2C^2}}$$

which was to be shown.

26-38

Since, with a given amplitude of driving voltage, the current amplitude at driving frequency ω_1 equals that at driving frequency ω_2, the impedance $Z_1 \equiv Z(\omega_1)$ must equal $Z_2 \equiv Z(\omega_2)$. For a series LRC circuit, the impedance $Z(\omega) = \sqrt{R^2 + [\omega L - (1/\omega C)]^2}$, so we have

$$\left(\omega_1 L - \frac{1}{\omega_1 C}\right)^2 = \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2 \quad (1)$$

or

$$\omega_1^2 L^2 - \frac{2L}{C} + \frac{1}{\omega_1^2 C^2} = \omega_2^2 L^2 - \frac{2L}{C} + \frac{1}{\omega_2^2 C^2}$$

or

$$L^2(\omega_1^2 - \omega_2^2) = \frac{1}{C^2}\left(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right) = \frac{(\omega_1^2 - \omega_2^2)}{C^2 \omega_1^2 \omega_2^2} \quad (2)$$

Equation (2) can be satisfied only if $\omega_1^2 = \omega_2^2$ (contrary to the information given in the exercise statement), or if

$$\omega_1^2 \omega_2^2 = \frac{1}{L^2 C^2} \quad (3)$$

According to Eq. (26-58), the resonant frequency (of a series LRC circuit is $\omega_r = 1/\sqrt{LC}$, so eq. (3) implies that

$$\omega_1 \omega_2 = \frac{1}{LC} = \omega_r^2$$

as desired.

<u>26-39</u>

We refer to Fig. 26E-39. When the bridge is balanced so that $i = 0$, there is no "communication" between the circuit branch DAE and the branch DBE. Hence the current in each branch is precisely what it would be if the other branch were not present. We adopt the following notations and sign conventions. The currents in branches DAE and DBE are positive-rightward. The generator voltage is positive when the electric potential is higher at D than at E. We use $V_{AD}(t)$ to denote the instantaneous value of the voltage drop <u>from D to A</u>. That is, $V_{AD}(t) > 0$ when the electric potential is higher at D than at A. Similarly, $V_{BD}(t)$ is the instantaneous value of the voltage drop <u>from D to B</u>. (Note these sign conventions carefully -- they are opposite from what the subscripts might normally indicate.)

(a) (i) When the bridge is balanced, the current in branch DAE has amplitude

$$\left|\vec{i}_1\right| = \frac{V_g}{Z_1} = \frac{V_g}{\sqrt{R_1^2 + (\omega L)^2}} \qquad (1)$$

where V_g is the amplitude of the voltage source. The first equality in (1) follows from the earlier remark about the effective isolation of each branch of a balanced bridge. The second equality, which uses $Z_1 = \sqrt{R_1^2 + (\omega L)^2}$, can be established by letting $C \to \infty$ in Eq. (26-74) for the impedance of an LRC series circuit. [In an ac circuit, a capacitor of very large capacitance is effectively a short circuit.]

(ii) The amplitude of the voltage across the inductor is the product of the current amplitude $|\vec{i}_1|$ and the inductor's impedance ωL:

$$\left|\vec{V}_{AD}\right| = \omega L\left|\vec{i}_1\right| = \frac{\omega L V_g}{\sqrt{R_1^2 + (\omega L)^2}} \qquad (2)$$

(iii) The current in the circuit path DBE has amplitude

$$\left|\vec{i}_2\right| = \frac{V_g}{Z_2} = \frac{V_g}{\sqrt{R_2^2 + (1/\omega C)^2}} \qquad (3)$$

The second equality in eq. (3) can be obtained by letting $L \to 0$ in Eq. (26-74). [In an ac circuit, an inductor of very small inductance is effectively a short circuit.]

(iv) The amplitude of the voltage across resistor 2 is the product of the current amplitude $|\vec{i}_2|$ and the resistance R_2:

$$\left|\vec{V}_{BD}\right| = R_2\left|\vec{i}_2\right| = \frac{R_2 V_g}{\sqrt{R_2^2 + (1/\omega C)^2}} \qquad (4)$$

(v) Since the ammeter presents negligible resistance to the flow of current, when the bridge is balanced the voltage $V_{AB}(t)$ across the ammeter must be zero. Kirchoff's voltage rule for the loop DAB then implies that $V_{AD}(t) = V_{BD}(t)$. In particular, the amplitudes are equal: $|\vec{V}_{AD}| = |\vec{V}_{BD}|$. Using eqs. (2) and (4), we have

$$\frac{\omega L V_g}{\sqrt{R_1^2 + (\omega L)^2}} = \frac{R_2 V_g}{\sqrt{R_2^2 + (1/\omega C)^2}}$$

or

$$\frac{R_2^2 + (1/\omega C)^2}{R_2^2} = \frac{R_1^2 + (\omega L)^2}{(\omega L)^2}$$

or

$$1 + \left(\frac{1}{\omega R_2 C}\right)^2 = 1 + \left(\frac{R_1}{\omega L}\right)^2 \qquad (5)$$

Since R_1, R_2, L, and C are all positive, eq. (5) implies that $1/(R_2C) = R_1/L$, or $R_1R_2 = L/C$, as was to be shown.

(b) (i) Because the path DAE is an LR circuit driven by a voltage source, the current $i_1(t)$ lags the applied voltage by an angle

$$|\varphi_1| = \tan^{-1}(\omega L/R_1) \qquad (6)$$

[This can be established by letting $C \to \infty$ in Eq. (26-76b).] The voltage $V_{AD}(t)$ across the inductor leads the current by 90°. Therefore $V_{AD}(t)$ leads the applied voltage by $\alpha_1 = 90° - |\varphi_1|$. The tangent of this angle is

$$\tan\alpha_1 = \tan(90° - |\varphi_1|) = \cot(|\varphi_1|) = \frac{1}{\tan|\varphi_1|}$$

$$= \frac{1}{(\omega L/R_1)} = \frac{R_1}{\omega L} \qquad (7)$$

(ii) Because the path DBE is an RC circuit driven by a voltage source, the current $i_2(t)$ leads the applied voltage by an angle

$$|\varphi_2| = \tan^{-1}[(1/\omega C)/R_2] \qquad (8)$$

[This can be established by letting $L \to 0$ in Eq. (26-76b).] The voltage $V_{BD}(t)$ across resistor 2 is in phase with the current. Therefore the voltage $V_{BD}(t)$ leads the applied voltage by the angle $|\varphi_2|$, whose tangent is

$$\tan|\varphi_2| = \frac{1}{\omega R_2 C} \qquad (9)$$

(iii) As discussed in section (v) of part (a), when the bridge is balanced, the voltages $V_{AD}(t)$ and $V_{BD}(t)$ must be equal in both amplitude and phase. Hence eqs. (7) and (9) imply that

$$\frac{R_1}{\omega L} = \frac{1}{\omega R_2 C}$$

or that $R_1R_2 = L/C$, as was to be shown.

26-40

(a) Referring to Fig. 26E-40, we know the magnetic flux $\bar{\Phi}$ that links the path ABCDE includes an additional contribution from the region within the wire between A'B'C'D'E' and ABCDE. Using the right-hand rule, it can be determined that this contribution has the same sign as that from the hollow space within the primed path. [This involves the assumption stated explicitly at the beginning of part (e) below.] The magnetic flux Φ' that links the path A'B'C'D'E' is only the flux through the hollow space.

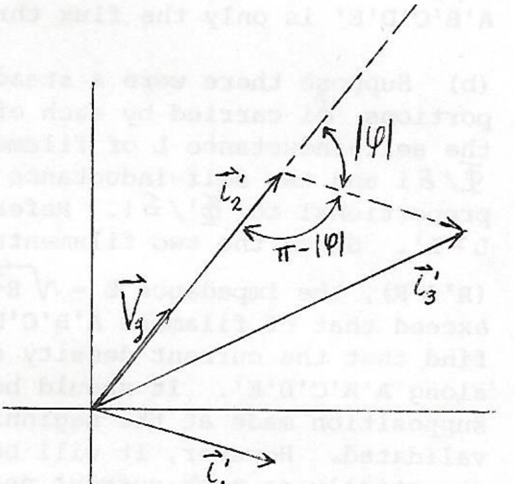
(b) Suppose there were a steady current i in the wire, with equal portions δi carried by each of the three filaments shown. Then the self-inductance L of filament ABCDE would be proportional to $\bar{\Phi}/\delta i$ and the self-inductance L' of the primed filament would be proportional to $\Phi'/\delta i$. Referring to part (a), we conclude that $L > L'$. Since the two filaments have approximately equal resistances ($R' \simeq R$), the impedance $Z = \sqrt{R^2 + (\omega L)^2}$ of filament ABCDE must exceed that of filament A'B'C'D'E'. [NOTE: In part (c) below, we find that the current density along ABCDE is smaller than that along A'B'C'D'E'. It should be acknowledged here that the supposition made at the beginning of part (b) would then be invalidated. However, it will be seen in part (e) that there is essentially as much current near the "outer" surface of the wire loop (at filament A"B"C"D"E") as there is near the "inner" surface (at A'B'C'D'E'). Hence the magnetic field between A'B'C'D'E' and ABCDE is indeed in the same direction as that within A'B'C'D'E', so that Φ exceeds Φ'. Hence for any given overall change in current through the loop of wire, there is a larger back emf along the unprimed filament than along the primed filament. And that is equivalent to saying that $L > L'$ and $Z > Z'$.]

(c) With any given applied ac voltage around the loop, the currents and current densities in filaments of equal cross section will be higher along A'B'C'D'E' (than along ABCDE), in the ratio Z/Z'. This is simply because the impedance to current flow is higher along ABCDE.

(d) The impedances are frequency-dependent. At high frequencies, the inductive reactance along each filament dominates the resistance, so that the ratio Z/Z' increases toward its high-frequency limit of L/L'.

(e) Assuming that as much current is carried around the loop by filaments between ABCDE and A"B"C"D"E" as between A'B'C'D'E' and ABCDE, the magnetic flux through the annulus between the unprimed and doubled-primed paths is the negative of the flux through the annulus between the primed and unprimed paths. This implies that the total flux Φ'' linking A"B"C"D"E" equals Φ', the flux in the hollow. Therefore, all results for the primed filament hold also for the double-primed one. In particular, there is more ac current carried along A"B"C"D"E" than along ABCDE. [NOTE: This conclusion actually must be used to vindicate the assumption at the beginning of part (e). Although the line of argument may seem to lie dangerously close to circular reasoning, the results obtained here are self-consistent. The reader can show that other a priori assumptions do not lead to self-consistency. In summary, the intent of the exercise is the presentation of a qualitative, intuitive discussion of the skin effect, not a full and formal mathematical approach.]

26-41



(a) Referring to Fig. 26E-41, we denote the current phasors in the inductor, the resistor, and the lead wire by $\vec{i}_1'$, $\vec{i}_2'$, and $\vec{i}_3'$, respectively. Since the currents are sinusoidal, these phasors are related to the rms currents by $|\vec{i}_1'| = i_1\sqrt{2}$, $|\vec{i}_2'| = i_2\sqrt{2}$, and $|\vec{i}_3'| = i_3\sqrt{2}$. Denoting the driving voltage by $V_g(t)$ (with corresponding phasor $\vec{V}_g$), we recognize that $\vec{i}_1'$ must lag $\vec{V}_g$ by the angle

$$\varphi = \tan^{-1} \omega L / \sqrt{R_L^2 + (\omega L)^2}$$

The current $\vec{i}_2'$ must be in phase with $\vec{V}_g$, and the phasor $\vec{i}_3' = \vec{i}_1' + \vec{i}_2'$. These relationships are exhibited in the phasor diagram.

(b) Applying the law of cosines to the phasor triangle in the diagram, we have

$$\left|\vec{i}_3\right|^2 = \left|\vec{i}_1\right|^2 + \left|\vec{i}_2\right|^2 - 2\left|\vec{i}_1\right|\cdot\left|\vec{i}_2\right|\cdot \cos(\pi - |\varphi|)$$

$$= \left|\vec{i}_1\right|^2 + \left|\vec{i}_2\right|^2 + 2\left|\vec{i}_1\right|\cdot\left|\vec{i}_2\right|\cdot \cos\varphi \qquad (1)$$

Using the fact that each phasor magnitude is just a factor of $\sqrt{2}$ larger than the corresponding rms current, we find that eq. (1) implies

$$i_3^2 = i_1^2 + i_2^2 + 2 i_2 i_1 \cos\varphi \qquad (2)$$

as desired.

(c) Since the rms voltage across the circuit is $V = i_2 R$, and the mean power dissipated in the inductor is $\langle P \rangle = V i_1 \cos\varphi$, we find

$$\langle P \rangle = i_2 R i_1 \cos\varphi = R(i_2 i_1 \cos\varphi) \qquad (3)$$

Using eq. (2), this can be rewritten as

$$P = \frac{1}{2} R(i_3^2 - i_2^2 - i_1^2)$$

which was to be shown.

CHAPTER TWENTY-SEVEN

27-3

Referring to Fig. 27-4 and to Eq. (27-4), we see that the left side of the equation has a definite value that depends only on the closed bounding curve and not at all on the bounded surface. If the right side of Eq. (27-4) had a value that was dependent on r_1, then evidently the equation could not be satisfied for all $r_1 < R$. That is, Maxwell's generalization of Ampere's law could not be valid -- it would be internally inconsistent.

27-5

The exercise statement clearly intends that we assume the hypothetically discovered "magnetic charge" to be the closest possible analogue of electric charge. Then there are two types of magnetic charges, "positive" and "negative". The magnetostatic analogue of Coulomb's law involves an inverse-square force, repulsive between like magnetic charges and attractive between unlike magnetic charges. These features can be accommodated by a law under which the force on a stationary particle ($\vec{v} = \vec{0}$) in a purely magnetic field ($\vec{\mathcal{E}} = \vec{0}$) is given by

$$\vec{F} = q_m \vec{\mathcal{B}} \qquad (1)$$

where $\vec{\mathcal{B}}$ is the (unique overall) magnetic field, including all sources (magnetic charges and electric currents). The contribution of magnetic charges to the magnetic field is described by a generalization of Maxwell's second equation, Eq. (27-6b):

$$\int_{\substack{\text{closed}\\\text{surface}}} \vec{\mathcal{B}} \cdot d\vec{a} = \frac{1}{\beta_0} \int_{\substack{\text{enclosed}\\\text{volume}}} \rho_m \, dv \qquad (2)$$