at infinity. That is, $|(\varphi')_o| = h/f_o$, as already stated. For a thick double-convex lens whose two surfaces have comparable curvatures, the criterion yields $|\varphi'| \simeq h/(s + \frac{1}{2}t)$ where t is the thickness of the lens and s is the distance (from the object to the vertex plane) for which $s' \rightarrow \infty$. In the specific case of the lens of Example 29-9, we use eq. (1) to find $s = -d/c = -(0.8667)/(-0.1433) = 6.05$ cm. Since t = 2.00 cm, we have $s + \frac{1}{2}t = 7.05$ cm. Thus we should expect

$$\frac{(\varphi'/\varphi)_o}{(\varphi'/\varphi)} \quad \frac{(h/f_o)}{[h/(s+\frac{1}{2}t)]} = \frac{7.05}{6.67} = 1.06$$

This value compares favorably with the result actually obtained:

$$\frac{(\varphi'/\varphi)_o}{(\varphi'/\varphi)} = \frac{3.75}{3.58} = 1.05$$

To put this another way, we refer to eq. (2), which gives the exact thick-lens result as $|\varphi'| = |ch| = h/|(1/c)| = h/|[1/(-0.1433)]| = h/(6.98 \text{ cm})$. The distance 6.98 cm is quite close to our estimate of 7.05 cm. In summary, a thick lens gives a smaller angular magnification than a thin one (with the same index and curvatures) because when each is positioned to give an image at infinity, the "effective center" of the thick lens is farther from the object (which therefore subtends a smaller angle $|\varphi'|$).

CHAPTER THIRTY

30-6

If the wavelength $\lambda = \lambda_C$, then the frequency $\nu = c/\lambda_C$, and the photon energy $E = h\nu = hc/\lambda_C$. Since $\lambda_C = h/m_e c$, we find

$$E = \frac{hc}{(h/m_e c)} = m_e c^2$$

which is the rest mass energy of the electron.

30-11

A position measurement of uncertainty Δx implies an uncertainty Δp_x in the x-component of momentum which satisfies

$$\Delta p_x \geqslant \frac{h}{4\pi \Delta x} \qquad (1)$$

Since the momentum and velocity of an electron are related by $\vec{p} = m_e\vec{v}$ (provided that $v << c$), inequality (1) implies a minimum uncertainty in the x-component of velocity:

$$(\Delta v_x)_{min} = \frac{(\Delta p_x)_{min}}{m_e} = \frac{h}{4\pi m_e \Delta x} \qquad (2)$$

With $\Delta x = 2 \times 10^{-11}$ m, eq. (2) yields

$$(\Delta v_x)_{min} = \frac{6.63 \times 10^{-34}\ \text{J}\cdot\text{s}}{4\pi(9.11 \times 10^{-31}\ \text{kg})(2 \times 10^{-11}\ \text{m})}$$

$$= \underline{2.9 \times 10^{6}\ \text{m/s}}$$

30-12

A position measurement of uncertainty Δx implies that the uncertainty in the x component of momentum must equal or exceed

$$(\Delta p)_{min} = \frac{h}{4\pi \Delta x}$$

For a particle moving nonrelativistically, the corresponding minimum uncertainty in the x component of velocity is given by

$$(\Delta v)_{min} = \frac{(\Delta p)_{min}}{m} = \frac{h}{4\pi m \Delta x}$$

If the position uncertainty $\Delta x = \lambda$, then

$$(\Delta v)_{min} = \frac{h}{4\pi m \lambda} = \frac{p}{4\pi m} = \frac{v}{4\pi}$$

which is comparable to the particle's speed v.

30-13

(a) According to the time-energy uncertainty principle, the energy uncertainty cannot be less than

$$(\Delta E)_{min} = \frac{h}{4\pi \Delta t}$$

where Δt is the duration of the energy measurement. For an unstable particle, Δt is limited to the mean life τ, so that the particle's total energy cannot be determined with an uncertainty smaller than

$$(\Delta E)_{min} = \frac{h}{4\pi \tau}$$

The mean life of a free neutron is $\tau = 930$ s, from which we find

$$(\Delta E)_{min} = \frac{(6.63 \times 10^{-34}\ J\cdot s)}{4\pi(930\ s)} = \underline{5.7 \times 10^{-38}\ J}$$

$$= \underline{3.5 \times 10^{-19}\ eV}$$

(b) For a neutron moving with negligible kinetic energy, the total energy equals the rest mass energy, so our answer must be the same as in part (a): $[\Delta(mc^2)]_{min} = \underline{5.7 \times 10^{-38}\ J} = \underline{3.5 \times 10^{-19}\ eV}$.

(c) The tabulated value for the rest mass of the neutron is m = 1.67×10^{-27} kg. Hence the fractional uncertainty implied by the result of part (b) is

$$\left(\frac{\Delta m}{m}\right)_{min} = \frac{[\Delta(mc^2)]_{min}}{mc^2} = \frac{5.7 \times 10^{-38}}{(1.67 \times 10^{-27})(3.00 \times 10^8)^2}$$

$$= \underline{3.8 \times 10^{-28}}$$

30-14

(a) At distance r from the light source, the power flux S is given by $S = P/4\pi r^2$, where P is the bulb power. With P = 10 W and r = 1.0 m, we find $S = 10/4\pi = 0.796\ W/m^2$. Each atom presents a cross sectional area πR^2, where $R = 1.0 \times 10^{-10}$ m. Under the stated assumptions, each atom absorbs energy at the steady rate

$$\frac{dE}{dt} = \pi R^2 S = \pi(1.0 \times 10^{-10}\ m)^2 (0.796\ W/m^2)$$

$$= \underline{2.5 \times 10^{-20}\ J/s = 0.156\ eV/s}$$

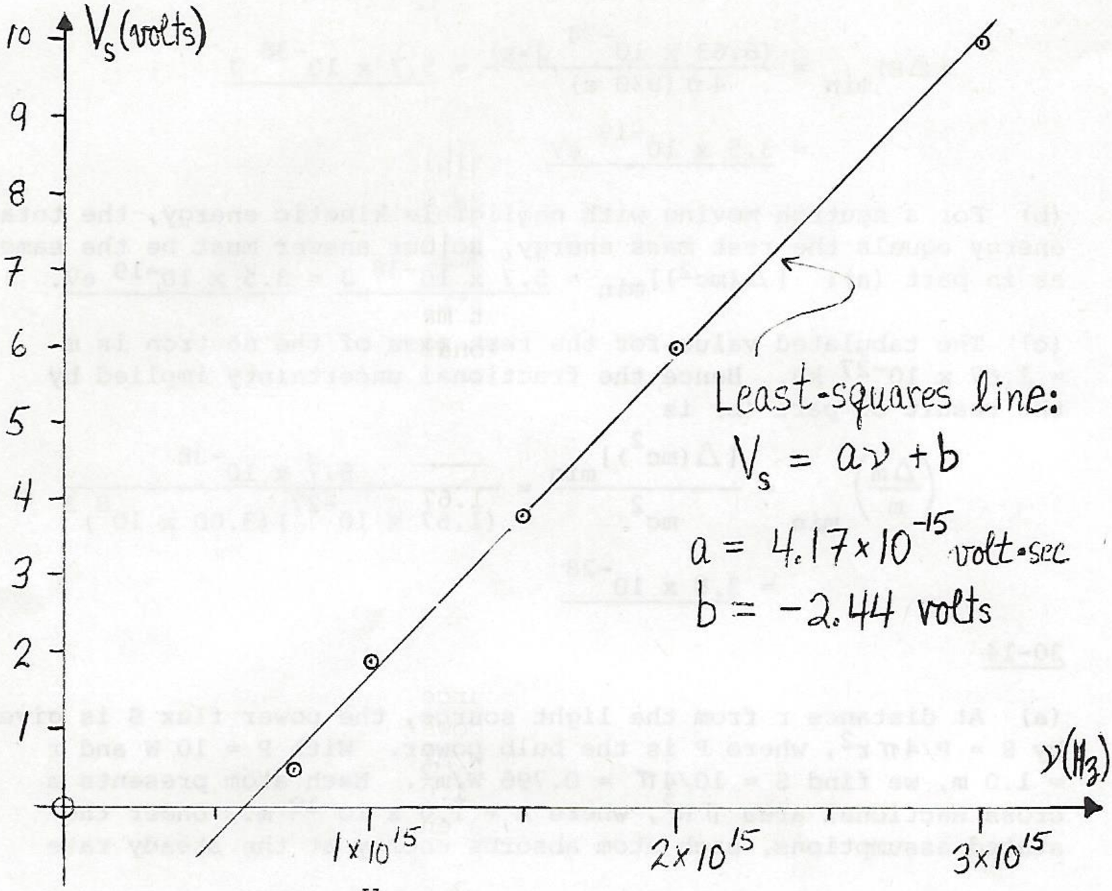
(b) Under the stated assumptions, the time required is given by

$$t = \frac{E}{(dE/dt)} = \frac{2.2\ eV}{(0.156\ eV/s)} = \underline{14.1\ s}$$

30-15

In solving this exercise, it is convenient to refer to a graph of the stopping potential versus the frequency of the incident light, as in Fig. 30-3. Such a graph is shown at the top of the next page. A least-squares straight line has been determined from the data and is indicated. [NOTE: We do not present here the details of least-squares fitting. A readable introduction can be found in the book Statistical Treatment of Experimental Data, by Hugh D. Young (McGraw-Hill Book Co., 1962)]. From the least-squares analysis, the "best" line is given by $V_s = a\nu + b$, with $a = 4.17 \times 10^{-15}$ V·s and b = -2.44 V.

(a) As discussed in Sec. 30-2 and summarized in Eq. (30-4), the stopping voltage vs. frequency relationship is



$$V_s = \left(\frac{h}{e}\right)\nu - \frac{W_{min}}{e}$$

Comparing this with the fitted line, we obtain h = ea = $(1.60 \times 10^{-19}$ J/V$)(4.17 \times 10^{-15}$ V s$) = 6.67 \times 10^{-34}$ J·s as the value of Planck's constant implied by the data.

(b) Since $b = -(W_{min}/e)$, the data imply a work function W_{min} = $-eb$ = <u>2.44 eV</u>.

(c) The cutoff frequency $\nu_c = W_{min}/h$. Using $h = 6.67 \times 10^{-34}$ J·s and W_{min} = 2.44 eV we obtain

$$\nu_c = \frac{(2.44)(1.60 \times 10^{-19}\ \text{J})}{(6.67 \times 10^{-34}\ \text{J·s})} = \underline{5.85 \times 10^{14}\ \text{Hz}}$$

(The cutoff frequency can also be read from the graph as the ν intercept.) The wavelength corresponding to ν_c is

$$\lambda_c = \frac{c}{\nu_c} = \frac{3.00 \times 10^8\ \text{m/s}}{5.85 \times 10^{14}\ \text{Hz}} = 5.13 \times 10^{-7}\ \text{m} = \underline{513\ \text{nm}}$$

(d) The stopping voltage $V_s = (h/e)\nu - (W_{min}/e) = a\nu + b$. For a wavelength λ = 50 nm, we have $\nu = c/\lambda = 6.0 \times 10^{15}$ Hz. Using the least-squares values for a and b, we obtain

$$V_s = (4.17 \times 10^{-15})(6.0 \times 10^{15}) - 2.44$$

$$= 25.0 - 2.44 = \underline{22.6 \text{ V}}$$

30-16

(a) A spherical object of density ρ and diameter d has mass

$$m = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 \cdot \rho = \frac{\pi d^3 \rho}{6}$$

To elevate such an object by its own diameter requires an energy investment of

$$mgd = \frac{\pi d^4 \rho g}{6}$$

Setting this equal to a given photon energy E, and solving for d, we find

$$d = \left(\frac{6E}{\pi \rho g}\right)^{1/4}$$

With E = 5000 eV = 8.0×10^{-16} J, $\rho = 3.0 \times 10^3$ kg/m^3, and g = 9.80 m/s^2, the sphere diameter is

$$d = \left[\frac{6(8.0 \times 10^{-16})}{\pi(3.0 \times 10^3)(9.80)}\right]^{1/4} = (5.20 \times 10^{-20})^{1/4}$$

$$= \underline{1.5 \times 10^{-5} \text{ m}}$$

(b) We find d/(0.5 mm) = $(1.5 \times 10^{-5})/(0.5 \times 10^{-3}) = 0.03$. That is, d is only about one-thirtieth the diameter of a typical grain of sand.

30-17

(a) The rate at which a flame of power P emits photons is

$$\frac{dN}{dt} = \frac{P}{\langle \epsilon \rangle}$$

where $\langle \epsilon \rangle$ is the average energy per photon. With P = 1.0×10^{-4} W and $\langle \epsilon \rangle$ = 2.0 eV = 3.2×10^{-19} J, we find

$$\frac{dN}{dt} = \frac{1.0 \times 10^{-4}}{3.2 \times 10^{-19}} = \underline{3.1 \times 10^{14} \text{ photons/s}}$$

(b) At a distance R from the flame, the number of photons per second which cross a unit area perpendicular to the line of sight is $(dN/dt)/4\pi R^2$. Hence the number of photons entering a circular opening of diameter d is given by

$$\frac{dn}{dt} = \left(\frac{1}{4\pi R^2}\frac{dN}{dt}\right)\left(\frac{\pi d^2}{4}\right) = \frac{d^2}{16R^2}\frac{dN}{dt}$$

With R = 10 m, d = 0.60 cm = 6.0×10^{-3} m, and dN/dt = 3.1×10^{14} photons/s, this yields

$$\frac{dn}{dt} = \frac{(6.0 \times 10^{-3})^2(3.1 \times 10^{14})}{16(10)^2} = \underline{7.0 \times 10^6 \text{ photons/s}}$$

(c) With R = 1.5 km = 1.5×10^3 m, d = 6.0×10^{-3} m, and dN/dt = 3.1×10^{14} photons/s, the equation for dn/dt yields

$$\frac{dn}{dt} = \frac{(6.0 \times 10^{-3})^2(3.1 \times 10^{14})}{16(1.5 \times 10^3)^2} = \underline{310 \text{ photons/s}}$$

30-18

An electron whose kinetic energy K_e = 100 keV has total energy $E_e = K_e + m_ec^2$ = 100 keV + 511 keV = 611 keV. Using the relativistic energy-momentum relationship

$$E^2 = (pc)^2 + (m_oc^2)^2$$

we find that a 100-keV electron has

$$p_ec = \sqrt{E_e^2 - (m_ec^2)^2} = \sqrt{(611)^2 - (511)^2} = 335 \text{ keV}$$

Therefore a photon with the same momentum as this electron has $p_{ph} = p_e$ = (335 keV)/c, so its energy is $E_{ph} = cp_e$ = $\underline{335 \text{ keV}}$.

30-19

(a) According to Eq. (30-14), when incident photons of wavelength λ are scattered through an angle φ by electrons, the photon wavelength after the scattering is λ', where

$$\lambda' - \lambda = \frac{h}{m_oc}(1 - \cos\varphi) \qquad (1)$$

Since $\lambda = c/\nu = hc/h\nu = hc/E$ and $\lambda' = hc/E'$, eq. (1) implies that

$$\frac{hc}{E'} - \frac{hc}{E} = \frac{h}{m_oc}(1 - \cos\varphi) \qquad (2)$$

Dividing eq. (2) by hc yields the desired result:

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_o c^2}(1 - \cos\varphi) \qquad (3)$$

(b) With $\varphi = 180°$, eq. (3) yields

$$\frac{1}{E'} - \frac{1}{E} = \frac{2}{m_o c^2} \qquad (4)$$

If $E - E' = \frac{1}{2}E$, then $E' = \frac{1}{2}E$, so eq. (4) becomes

$$\frac{2}{E} - \frac{1}{E} = \frac{2}{m_o c^2}$$

which implies $E = \underline{\frac{1}{2}m_o c^2} = \frac{1}{2}(0.511 \text{ MeV}) = \underline{0.256 \text{ MeV}}$.

30-20

(a) One of the basic results of plane geometry is that any angle inscribed in a semicircle is a right angle. For every photon that is emitted at S and scattered to the detector at D, the path forms an angle inscribed in a semicircle. Hence the scattering angle is 90° for every such photon. (NOTE: In making this statement, we neglect the possibility that photons may undergo two or more scatterings in traveling from S to D. That is, we ignore "multiple scattering.") The advantage of the apparatus shown in Fig. 30E-20, compared to that shown in Fig. 30-5, is that the barrier of Fig. 30E-20 "wastes" fewer of the source photons than does the first set of collimators in Fig. 30-5. In other words, the scattering material subtends a greater solid angle, as viewed from the source.

(b) According to Eq. (30-14), we have

$$\lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\varphi) \qquad (1)$$

Since $\lambda = c/\nu = hc/h\nu = hc/E$ and $\lambda' = hc/E'$, eq. (1) can be rewritten as

$$\frac{hc}{E'} - \frac{hc}{E} = \frac{h}{m_o c}(1 - \cos\varphi)$$

or as

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_o c^2}(1 - \cos\varphi) \qquad (2)$$

With $\varphi = 90°$, eq. (2) yields

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_o c^2}$$

so that

$$E' = \frac{E(m_o c^2)}{E + m_o c^2} \qquad (3)$$

With E = 1.00 MeV and $m_o c^2$ = 0.511 MeV, eq. (3) implies that E' = (1.00)(0.511)/(1.511) = 0.34 MeV.

30-21

We begin with Eq. (30-14), which relates the initial and final wavelengths of the scattered photon:

$$\lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\varphi) \qquad (1)$$

Since $\lambda = c/\nu = hc/h\nu = hc/E$ and $\lambda' = hc/E'$, we find

$$\frac{hc}{E'} - \frac{hc}{E} = \frac{h}{m_o c}(1 - \cos\varphi)$$

or

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_o c^2}(1 - \cos\varphi) \qquad (2)$$

Solving for E', we find

$$E' = \left[\frac{1}{E} + \frac{(1 - \cos\varphi)}{m_o c^2}\right]^{-1}$$

$$= \frac{E \cdot m_o c^2}{[m_o c^2 + E(1 - \cos\varphi)]} = \frac{E}{[1 + (E/m_o c^2)(1 - \cos\varphi)]} \qquad (3)$$

Since the total energy of the electron-photon system is conserved, and since the electron has negligible kinetic energy before the scattering event, the electron's final kinetic energy K = E - E'. Using eq. (3), we obtain

$$K = E - \frac{E}{[1 + (E/m_oc^2)(1 - \cos\varphi)]}$$

$$= \frac{E + (E^2/m_oc^2)(1 - \cos\varphi) - E}{[1 + (E/m_oc^2)(1 - \cos\varphi)]}$$

$$= \frac{(E^2/m_oc^2)(1 - \cos\varphi)}{1 + (E/m_oc^2)(1 - \cos\varphi)}$$

which was to be shown. [NOTE: In the first printing of the text, the fraction in the denominator was incorrectly given as (E^2/m_oc^2).]

30-22

(a) The flight time is equal to the path length L divided by the speed of light. With L = 1.0 m, we find t = (1.0 m)/(3.00×10^8 m/s) = 3.33×10^{-9} s.

(b) According to the hint, the fraction of photons whose flights overlap is given by f = 2Rt, where R is the number of photons per unit time. In order to keep 1 - f > 0.99, as desired, we must have 2Rt < 0.01, so that $R < R_{max}$ = 0.01/2t = 0.01/(6.67×10^{-9} s) = 1.5×10^6 photons/s.

(c) Since the rate is limited to R_{max}, the minimum exposure time required to accumulate N photons is given by $t_e = N/R_{max}$. With N = 1.0×10^8 photons and R_{max} = 1.5×10^6 photons/s, we obtain $t_e = (1.0 \times 10^8)/(1.5 \times 10^6)$ = 67 s.

30-23

(a) An object of mass m which falls a distance z from rest has kinetic energy $p^2/2m = mgz$. The corresponding momentum is p = $m\sqrt{2gz}$, so the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{m\sqrt{2gz}} \qquad (1)$$

If the object is a sphere of density ρ and diameter d, its mass is

$$m = \frac{4\pi}{3}\left(\frac{d}{2}\right)^3\rho = \frac{\pi\rho d^3}{6} \qquad (2)$$

Therefore, after falling a distance z equal to its diameter, its de Broglie wavelength is

$$\lambda = \frac{h}{(\pi\rho d^3/6)\sqrt{2gd}} = \frac{6h}{\pi\rho d^{7/2}\sqrt{2g}} \qquad (3)$$

If its de Broglie equals its diameter, eq. (3) becomes

$$d = \frac{6h}{\pi\rho d^{7/2}\sqrt{2g}}$$

Solving for the diameter, we find

$$d = \left(\frac{6h}{\pi\rho\sqrt{2g}}\right)^{2/9} \qquad (4)$$

(b) With $\rho = 1.00\ g/cm^3 = 1.00 \times 10^3\ kg/m^3$, $g = 9.80\ m/s^2$, and $h = 6.63 \times 10^{-34}\ J\cdot s$, eq. (4) yields

$$d = \left[\frac{6(6.63 \times 10^{-34}\ kg\cdot m^2/s)}{\pi(1.00 \times 10^3\ kg/m^3)\sqrt{2}\,(\sqrt{9.80}\ m^{1/2}/s)}\right]$$

$$= (2.860 \times 10^{-37})^{2/9}(m^{9/2})^{2/9}$$

$$= \underline{7.57 \times 10^{-9}\ m = 7.57\ nm}$$

This is about two orders of magnitude smaller than visible wavelengths, so we could not expect to see such an object.

30-24

The relativistic relationship between momentum and energy for an electron is

$$E = \sqrt{(cp)^2 + (m_e c^2)^2} \qquad (1)$$

Since $p = h/\lambda$, $pc = hc/\lambda$. If the de Broglie wavelength λ is required to be no larger than 1.00 fermi, we have

$$pc = \frac{hc}{\lambda} \geq \frac{(6.63 \times 10^{-34})(3.00 \times 10^8)}{(1.00 \times 10^{-15})}$$

$$= 1.99 \times 10^{-10}\ J = 1.24 \times 10^9\ eV$$

$$= 1.24 \times 10^3\ MeV \qquad (2)$$

(NOTE: 1 MeV = 10^6 eV; 1 GeV = 10^9 eV = 10^3 MeV) With $m_e c^2$ = 0.511 MeV, we can use inequality (2) in equation (1) to obtain

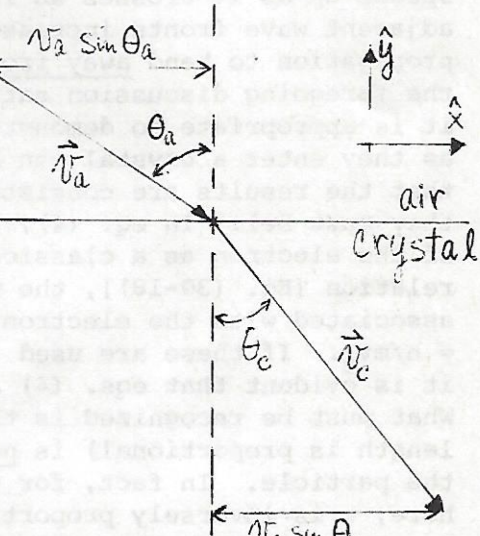
$$E \geqslant \sqrt{(1240)^2 + (0.511)^2} = 1.24 \times 10^3 \text{ MeV}$$

Hence the electron's kinetic energy K must satisfy

$$K = E - m_e c^2 \geqslant 1.24 \times 10^3 - 0.511 = 1240 \text{ MeV}$$

$$= \underline{1.24 \text{ GeV}} = \underline{1.99 \times 10^{-10} \text{ J}}$$

30-25



Because of the attraction of the crystal for the electron, the electron's speed increases as it enters the crystal. Since the force which the crystal exerts on the electron (as it crosses the interface) is perpendicular to the interface (and since the electron's motion is assumed to be nonrelativistic), there is no change in the component of the electron's velocity parallel to the interface. Using the notation indicated in the figure at the right, we have

$$\vec{v}_c \cdot \hat{x} = \vec{v}_a \cdot \hat{x} \qquad (1)$$

But the diagram shows that

$$\vec{v}_c \cdot \hat{x} = v_c \sin\theta_c \qquad (2)$$

and

$$\vec{v}_a \cdot \hat{x} = v_a \sin\theta_a \qquad (3)$$

Equations (1) - (3) imply that

$$v_c \sin\theta_c = v_a \sin\theta_a \qquad (4)$$

as was to be shown. We now discuss the relationship between equation (4) and Snell's law for the refraction of a wave. When Eqs. (28-4a), (28-4b), and (28-6) are generalized to describe a wave whose speed changes from w to w', Snell's law takes the form

$$\frac{\sin\theta'}{\lambda'} = \frac{\sin\theta}{\lambda} \qquad (5)$$

when written in terms of the wavelengths in the two media. [Eq.

(5) can be derived from equations analogous to Eqs. (28-4a) and (28-4b), and a figure analogous to Fig. 28-16, using a time interval T chosen to equal one wave period, so that $\ell = \lambda$ and $\ell' = \lambda'$.] We observe that if $w' > w$, so that $\lambda' > \lambda$, then $\theta' > \theta$. The difference between the law of refraction for particles, as expressed in eq. (4), and that for waves, as expressed in eq. (5), can be detailed as follows. When a particle speeds up as it crosses an interface, its path is bent toward the normal line. When a wave speeds up as it crosses an interface, the separation between adjacent wave fronts increases. This requires the direction of propagation to bend away from the normal line. [NOTE: Although the foregoing discussion satisfies the requirements of the exercise, it is appropriate to demonstrate that the refraction of electrons as they enter a crystal can be described as a wave phenomenon, and that the results are consistent with the particle description (as they must be). In eq. (4), the speeds v_a and v_c are the speeds of the electron as a classical particle. According to de Broglie's relation [Eq. (30-18)], the wavelengths λ_a and λ_c of the waves associated with the electron are $\lambda = \lambda_a = h/mv_a$ and $\lambda' = \lambda_c = h/mv_c$. If these are used in eq. (5) with $\theta = \theta_a$ and $\theta' = \theta_c$, it is evident that eqs. (4) and (5) are completely equivalent. What must be recognized is that the wave speed w (to which the wavelength is proportional) is not the same as the classical speed of the particle. In fact, for the nonrelativistic motions described here, w is inversely proportional to the classical particle speed v. It is left to the reader to show that since de Broglie relationship $p = h/\lambda$ is also valid for relativistic motions, eq. (5) will be consistent with the relativistic generalization of eq. (4), namely $p_c \sin\theta_c = p_a \sin\theta_a$.]

30-26

(a) Since $\Delta\nu\Delta t \geq 1/4\pi$, the duration Δt of a musical tone of nominal frequency ν_o satisfies

$$\Delta t \geq \frac{1}{4\pi\Delta\nu_{max}} = \frac{1}{4\pi(10^{-2}\nu_o)} = \frac{25}{\pi\nu_o}$$

(i) For $\nu_o = 4100$ Hz, we find $\Delta t \geq 25/(4100\pi) = 1.9 \times 10^{-3}$ s = 1.9 ms.

(ii) For $\nu_o = 440$ Hz, we find $\Delta t \geq 25/(440\pi) = 1.8 \times 10^{-2}$ s = 18 ms

(iii) For $\nu_o = 33$ Hz, we find $\Delta t \geq 25/(33\pi) = 0.24$ s.

(b) It is not possible to have a succession of musical-sounding low notes unless the duration of each note is at least a few tenths of a second.

30-27

(a) Because the pion's existence implies a violation of energy conservation by at least $\delta E = mc^2$ (where m is the pion mass), the duration of its existence cannot exceed

$$\Delta t = \frac{h/4\pi}{\delta E} = \frac{h}{4\pi mc^2} \qquad (1)$$

If the duration were longer than this, an experimental arrangement could be devised that would determine the energy of the system with an uncertainty smaller than δE, which would expose the failure of energy conservation. (Such failures have never been observed.)

(b) Presuming that the speed of light is the highest possible speed even for the motion of virtual events, the maximum distance that the pion can travel is $R_{max} = c\Delta t$. Using eq. (1) we find

$$R_{max} = \frac{h}{4\pi mc} \qquad (2)$$

As stated in the exercise, this provides an estimate for the range of the interaction. [NOTE: It can be shown that if virtual particles could travel with speeds greater than c, a basic feature of physics, known as "causality", would be violated. No such violations have been observed.]

(c) From eq. (2), we find

$$mc^2 \simeq \frac{hc}{4\pi R_{max}} \qquad (3)$$

Using the observed range $R_{max} \simeq 2 \times 10^{-15}$ m, we obtain an estimate for the rest mass energy:

$$mc^2 \simeq \frac{(6.63 \times 10^{-34})(3.00 \times 10^{8})}{4\pi(2 \times 10^{-15}\ \text{m})}$$

$$= 7.9 \times 10^{-12}\ \text{J} = \underline{50\ \text{MeV}}$$

This is of the same order of magnitude as 140 MeV.

30-28

(a) A reasonable expression for the angular width of the central maximum of the pattern is half the distance between the first-order minima. From Eq. (28-19), the first minima occur at $\pm\theta_1$, where

$$\sin\theta_1 = \frac{\lambda}{a} \qquad (1)$$

Here λ is the wavelength of the waves passing through the slit.

Using the de Broglie relation, we find the angular width to be

$$\frac{1}{2}[\theta_1 - (-\theta_1)] = \frac{1}{2}\left[2\sin^{-1}\left(\frac{\lambda}{a}\right)\right] = \sin^{-1}\left(\frac{h}{p_o a}\right) \qquad (2)$$

(b) The x component of the momentum of a particle traveling in the xy plane at angle θ with the positive y axis is $p_x = p_o \tan\theta$. Therefore particles in the central maximum have p_x values in the range $-p_o \tan\theta_1 < p_x < p_o \tan\theta_1$. Taking Δp_x to be one-half of this range, we have

$$\Delta p_x \simeq p_o \tan\theta_1 = p_o \tan\left[\sin^{-1}\left(\frac{h}{p_o a}\right)\right] \gtrsim \frac{h}{a} \qquad (3)$$

The last relationship in (3) needs to be justified. If $\theta_1 = h/p_o a \ll 1$, then $\tan\theta_1 \simeq \sin\theta_1 \simeq \theta_1$. But for any value of ξ in the range $0 < \xi \leq 1$, $\tan[\sin^{-1}(\xi)] > \xi$. This is easily proved. We observe that $\tan\theta = \sin\theta/\cos\theta = \sin\theta/\sqrt{1-\sin^2\theta}$. Therefore, if $\theta = \sin^{-1}(\xi)$, then $\tan\theta = \xi/\sqrt{1-\xi^2} > \xi$.

(c) Since the position uncertainty in the x coordinate of particles in the diffracted beam is at least $\frac{1}{2}a$, eq. (3) implies that the uncertainty product satisfies

$$\Delta x \Delta p_x \gtrsim \left(\frac{a}{2}\right)\left(\frac{h}{a}\right) = \frac{h}{2} > \frac{h}{4\pi}$$

The uncertainty principle is not violated.

30-29

(a) Since the energy density in the frequency range ν to $\nu + d\nu$ is

$$\mathcal{E}(\nu)d\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu \qquad (1)$$

The overall energy density must be

$$\mathcal{E}_T = \int_0^\infty \mathcal{E}(\nu)d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1} \qquad (2)$$

We now rewrite $\mathcal{E}_T$ in terms of a dimensionless variable $x \equiv h\nu/kT$. Since $\nu = (kT/h)x$, we have $d\nu = (kT/h)dx$, and eq. (2) becomes

$$\mathcal{E}_T = \frac{8\pi h}{c^3}\int_0^\infty \frac{(kT/h)^3 x^3 (kT/h)dx}{e^x - 1}$$

$$= \frac{8\pi (kT)^4}{h^3 c^3}\int_0^\infty \frac{x^3 dx}{e^x - 1} \qquad (3)$$

(NOTE: The reader should observe that for $\nu = 0$, $x = 0$ and for $\nu \to \infty$, $x \to \infty$.)

(b) In SI units, the quantity

$$\frac{8\pi k^4}{h^3 c^3} = \frac{8\pi(1.3807 \times 10^{-23})^4}{(6.626 \times 10^{-34})^3 (2.998 \times 10^8)^3}$$

$$= 1.165 \times 10^{-16} \ \mathrm{J/(m^3 K^4)} \qquad (4)$$

The integral I_3 in eq. (3) can be evaluated numerically, or an analytical method can be employed. [See, for example, Statistical and Thermal Physics, by F. Reif (McGraw-Hill Book Co., 1965)]. The analytical approach leads to the result

$$I_3 \equiv \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15} \qquad (5)$$

Using eqs. (4) and (5) in eq. (3), we find that

$$\epsilon_T = \frac{\pi^4}{15}(1.165 \times 10^{-16})T^4 = 7.57 \times 10^{-16} T^4 \qquad (6)$$

when ϵ_T is measured in $\mathrm{J/m^3}$ and T is measured in K. This is the desired result.

(c) The energy density $\epsilon(\nu)d\nu$ of radiation in the frequency range ν to $\nu + d\nu$ must be the product of the single-photon energy $E(\nu) = h\nu$ and the number density $n(\nu)d\nu$ of photons whose energy lies in the range $h\nu$ to $h\nu + hd\nu$:

$$\epsilon(\nu)d\nu = E(\nu)n(\nu)d\nu \qquad (7)$$

Solving for the number density and using eq. (1), we find

$$n(\nu)d\nu = \frac{\epsilon(\nu)d\nu}{E(\nu)} = \frac{1}{h\nu}\frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/kT} - 1}$$

$$= \frac{8\pi\nu^2}{c^3}\frac{1}{e^{h\nu/kT} - 1} d\nu \qquad (8)$$

(d) The overall photon density n_T is the integral of eq. (8) over all frequencies. Using the same dimensionless variable x that was employed in part (a), we find

$$n_T = \int_0^\infty n(\nu)d\nu = \frac{8\pi}{c^3}\int_0^\infty \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1}$$

$$= \frac{8\pi}{c^3}\int_0^\infty \frac{(kT/h)^2 x^2 (kT/h)dx}{e^x - 1}$$

......

$$= \frac{8\pi(kT)^3}{h^3c^3}\int_0^\infty \frac{x^2\,dx}{e^x - 1} \qquad (10)$$

as was to be shown.

(e) In SI units, the quantity

$$\frac{8\pi k^3}{h^3c^3} = \frac{8\pi(1.3807 \times 10^{-23})^3}{(6.626 \times 10^{-34})^3(2.998 \times 10^8)^3}$$

$$= 8.439 \times 10^6\ m^3/K^3 \qquad (11)$$

The value of the integral $I_2 \equiv \int_0^\infty [x^2/(e^x - 1)]dx$ in eq. (10) can be obtained by numerical integration, or it can be read from a table. [In fact, $I_2 = 2\zeta(3)$, where $\zeta(3)$ is the Riemann zeta function of argument 3.] The result is

$$I_2 \equiv \int_0^\infty \frac{x^2}{e^x - 1}\,dx = 2.404 \qquad (12)$$

Using eqs. (11) and (12) in eq. (10), we find that

$$n_T = (2.404)(8.439 \times 10^6)T^3 = 2.03 \times 10^7 T^3 \qquad (13)$$

when n_T is measured in photons/m^3 and T is measured in K. This is the desired result.

30-30

The average energy per photon is the overall energy density divided by the overall photon density:

$$\langle E \rangle = \frac{\epsilon_T}{n_T} = \frac{7.57 \times 10^{-16} T^4}{2.03 \times 10^7 T^3} = 3.73 \times 10^{-23}\ T \qquad (1)$$

where $\langle E \rangle$ is measured in J and T is measured in K. Since k = 1.3807×10^{-23} J/K, eq. (1) implies that

$$\langle E \rangle = \frac{3.73 \times 10^{-23}}{1.3807 \times 10^{-23}} kT = \underline{2.70\ kT} \qquad (2)$$

Notice that the coefficient 2.70 in eq. (2) is dimensionless, so the equation's validity does not depend on the use of SI units. The ratio of the average photon energy in thermal radiation at temperature T to the average kinetic energy of a molecule in a monatomic gas at the same temperature is given by

$$\frac{\langle E \rangle}{(3kT/2)} = \frac{2(2.70)}{3} = \underline{1.80}$$

30-31

We present two solutions for this exercise. In both solutions, we begin by assuming that it is possible for a free electron to absorb a photon.

(i) We analyze the collision using a reference frame in which the electron is at rest after the collision. Evidently this is a zero-momentum (or center-of-momentum) frame for the system. By the conservation of momentum of the system, the total momentum of the system is zero before as well as after the collision. Hence, prior to the collision, the electron and photon are approaching one another with equal but opposite momenta. Using primes to denote quantities measured in our zero-momentum frame, the total energy of the system is initially

$$E_i' = E_\gamma' + (m_e c^2 + K_e') > m_e c^2$$

(Only in the trivial case $E_\gamma' = 0$ does $E_i' = m_e c^2$). Here E_γ' is the energy of the photon and K_e' is the (initial) kinetic energy of the electron. However, the final state of the system consists of a lone electron at rest, so the final energy of the system is $E_f' = m_e c^2$. Therefore $E_f' < E_i'$: the law of conservation of energy is violated.

In summary, by assuming that momentum is conserved in this process, we have found that energy cannot be conserved. (If we had assumed that energy is conserved, we would have found a non-conservation of momentum.) Furthermore, in relativistic mechanics, the conservation laws for energy and momentum are inextricably linked. What appears as an expression purely of energy conservation (or non-conservation!) in one inertial frame involves both the energy and the momentum in some other frame. (This is examined in Exercise 15-32.) Hence the absorption of a photon by a free electron must violate the energy and/or momentum conservation laws in any inertial frame. Since these laws are believed to be universally valid, we conclude that the alleged process cannot take place.

(ii) We analyze the collision using an inertial frame in which the electron is initially at rest. Assuming momentum conservation, we know that the final momentum $\vec{p}_f$ of the electron must equal the momentum $\vec{p}_\gamma$ of the incident photon. We let p_γ denote the common magnitude $|\vec{p}_f| = |\vec{p}_\gamma|$. We now develop expressions for the energy of the system before and after the collision. The initial total energy $E_i = E_\gamma + m_e c^2$, where E_γ is the photon energy. Since $E_\gamma = cp_\gamma$, we have

$$E_i = cp_\gamma + m_e c^2 \tag{1}$$

The final state consists of a lone electron moving with a momentum of magnitude p_γ, so

$$E_f = \sqrt{(cp_\gamma)^2 + (m_e c^2)^2} \qquad (2)$$

Equations (1) and (2) imply that

$$E_i^2 = (cp_\gamma)^2 + 2(cp_\gamma)(m_e c^2) + (m_e c^2)^2$$

$$> (cp_\gamma)^2 + (m_e c^2)^2 = E_f^2 \qquad (3)$$

That is, energy is not conserved (except in the trivial case $p_\gamma = 0$).

At this point, the summarizing comments in the first solution could all be repeated.

30-32

According to Eq. (30-14), the wavelength change of the photon is

$$\lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\varphi)$$

Since $\lambda = c/\nu = hc/h\nu = hc/E$ and $\lambda' = hc/E'$, we find

$$\frac{hc}{E'} - \frac{hc}{E} = \frac{h}{m_o c}(1 - \cos\varphi)$$

or

$$\frac{1}{E'} = \frac{1}{E} + \frac{(1 - \cos\varphi)}{m_o c^2} \qquad (1)$$

Inverting eq. (1), we obtain

$$E' = \left[\frac{1}{E} + \frac{(1 - \cos\varphi)}{m_o c^2}\right]^{-1}$$

$$= \frac{m_o c^2}{(1 - \cos\varphi) + (m_o c^2/E)} \qquad (2)$$

For a scattering angle $\varphi > 60°$, $\cos\varphi < 1/2$. Hence $(1 - \cos\varphi) > 1/2$, so eq. (2) implies

$$E' < \frac{m_o c^2}{\frac{1}{2} + (m_o c^2/E)} = \frac{2m_o c^2}{1 + (2m_o c^2/E)} < 2m_o c^2$$

for any $E > 0$. This is the desired result.

30-33

(a) As discussed in Chapter 23, an electron executes circular motion when it is given an initial velocity $\vec{v}_i$ at right angles to a uniform magnetic field $\mathcal{B}$. The relativistically correct version of Newton's second law gives

$$\left|\frac{d\vec{p}_e}{dt}\right| = \omega p_e = \left|\vec{F}\right| = e\mathcal{B}v \qquad (1)$$

in this case. Here ω is the angular speed of the circular motion. Since v/ω equals the radius R of the orbit, eq. (1) yields

$$p_e = e\mathcal{B}R \qquad (2)$$

[NOTE: A more complete justification of eq. (2) is given in the solution for Exercise 23-20.] Employing eq. (2) in the relativistic energy-momentum relationship, we find that the electron's total energy is

$$E_e = \sqrt{(cp_e)^2 + (m_o c^2)^2} = \sqrt{(e\mathcal{B}Rc)^2 + (m_o c^2)^2} \qquad (3)$$

With $\mathcal{B} = 0.150$ T and R = 3.00 cm = 3.00×10^{-2} m, we find

$$e\mathcal{B}Rc = (1.60 \times 10^{-19})(0.15)(3.00 \times 10^{-2})(3.00 \times 10^{8})$$

$$= 2.16 \times 10^{-13} \text{ J} = 1.35 \text{ MeV}$$

Using this in eq. (3) with $m_e c^2 = 0.511$ MeV, we obtain

$$E_e = \sqrt{(1.35)^2 + (0.511)^2} = \underline{1.44 \text{ MeV}} \qquad (4)$$

Hence the electron's kinetic energy is

$$K_e = E_e - m_o c^2 = 1.44 - 0.511 = \underline{0.93 \text{ MeV}} \qquad (5)$$

(b) According to Eq. (30-14), the wavelength change in Compton scattering is

$$\lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\varphi) \qquad (6)$$

Since $\lambda = c/\nu = hc/h\nu = hc/E$ and $\lambda' = hc/E'$, we find

$$\frac{1}{E'} - \frac{1}{E} = \frac{(1 - \cos\varphi)}{m_o c^2} \qquad (7)$$

For backward scattering ($\varphi = 180°$), eq. (7) becomes

$$\frac{1}{E'} - \frac{1}{E} = \frac{2}{m_o c^2}$$

or

$$\frac{E - E'}{EE'} = \frac{2}{m_o c^2} \qquad (8)$$

The conservation of energy, as applied to Compton scattering from an electron whose initial kinetic energy is negligible, implies that the initial photon energy is divided between the scattered photon and the struck electron:

$$E = E' + K_e \qquad (9)$$

Using eq. (9) to eliminate E' from eq. (8), we find

$$\frac{K_e}{E(E - K_e)} = \frac{2}{m_o c^2}$$

so that

$$E^2 - K_e E - \frac{1}{2} K_e (m_o c^2) = 0 \qquad (10)$$

The solutions of the quadratic equation (10) are:

$$E = \frac{K_e \pm \sqrt{K_e^2 + 2K_e (m_o c^2)}}{2}$$

Since E must be positive, we have

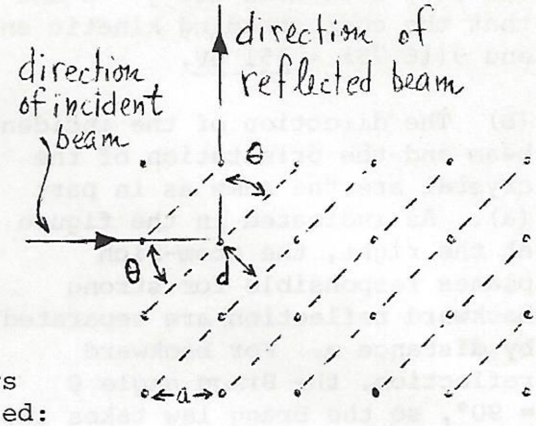
$$E = \frac{K_e + \sqrt{K_e^2 + 2K_e (m_o c^2)}}{2} \qquad (11)$$

With $K_e = 0.93$ MeV from eq. (5), we find that the incident photon had energy

$$E = \frac{0.93 + \sqrt{(0.93)^2 + 2(0.93)(0.511)}}{2} = \underline{1.14 \text{ MeV}}$$

[NOTE: The photon's final energy is $E' = E - K_e = 1.14 - 0.93 = 0.21$ MeV. The photon "lost" more than 80% of its initial energy in the process of being back-scattered.]

30-34



(a) Since the electrons are reflected through a right angle, the angles θ shown in the figure at the right equal 45°. Strong reflection is produced by the constructive superposition of electron waves diffracted by the indicated parallel atom-rich planes. The distance d between these planes equals $a/\sqrt{2}$. Constructive superposition occurs only when Bragg's law is satisfied:

$$j\lambda = 2d \sin\theta = 2(a/\sqrt{2}) \sin 45° = 2(a/\sqrt{2})(\sqrt{2}/2) = a$$

or when

$$\lambda = \lambda_j = a/j \qquad (1)$$

for j = 1,2,3,.... The corresponding momenta of the Bragg-reflected electrons are

$$p_j = \frac{h}{\lambda_j} = \frac{jh}{a} \qquad (2)$$

and their kinetic energies are

$$K_j = \frac{p_j^2}{2m} = \frac{j^2 h^2}{2ma^2} \qquad (3)$$

With $a = 3.00 \times 10^{-10}$ m, we find

$$\frac{h^2}{2ma^2} = \frac{(6.63 \times 10^{-34})^2}{2(9.11 \times 10^{-31})(3.00 \times 10^{-10})^2}$$

$$= 2.681 \times 10^{-18} \text{ J} = 16.75 \text{ eV}$$

Hence the energies for which the Bragg condition is satisfied are

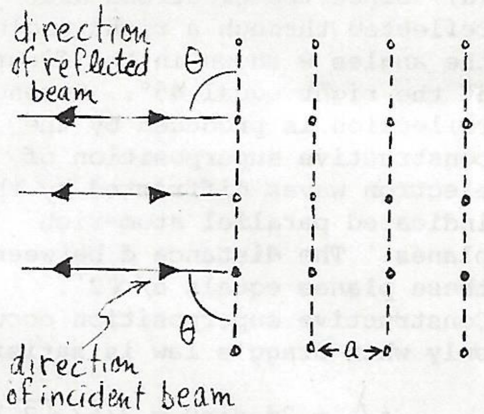
$$K_j = j^2 (16.75 \text{ eV}) \qquad (4)$$

for j = 1,2,3,.... Since the kinetic energies represented in the beam lie between 50 and 200 eV, we must have $50 < 16.75j^2 < 200$, or

$$1.73 < j < 3.46$$

The only solutions are j = 2 and j = 3. Using eq. (4), we find that the corresponding kinetic energies are 4(16.75) = 67.0 eV and 9(16.75) = 151 eV.

(b) The direction of the incident beam and the orientation of the crystal are the same as in part (a). As indicated in the figure at the right, the atom-rich planes responsible for strong backward reflection are separated by distance a. For backward reflection, the Bragg angle $\theta = 90°$, so the Bragg law takes the form



$$j\lambda = 2d \sin\theta = 2a \sin\theta = 2a$$

so that

$$\lambda = \lambda_j = \frac{2a}{j}$$

for j = 1,2,3,.... Since the wavelength λ_j here is twice that obtained in part (a), the momentum p_j must be halved, and the kinetic energy K_j must be quartered. Using eq. (4), we have

$$K_j = j^2[(16.75)/4] = j^2(4.19 \text{ eV}) \qquad (5)$$

Since the kinetic energies represented in the beam lie between 50 and 200 eV, we must have $50 < 4.19j^2 < 200$, or

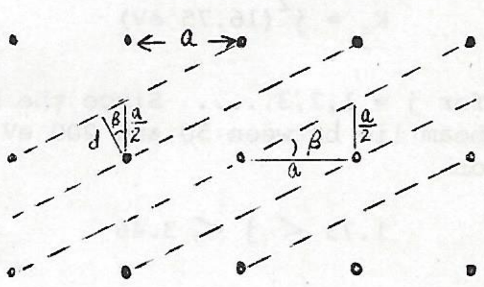
$$3.45 < j < 6.91$$

The solutions are j = 4, j = 5, and j = 6. Using eq. (5), the corresponding kinetic energies are 16(4.19) = 67.0 eV, 23(4.19) = 105 eV, and 36(4.19) = 151 eV.

30-35

(a) Referring to the diagram at the right, we observe that $\tan\beta = (a/2) \div a = 1/2$. Therefore

$$\beta = \tan^{-1}(\tfrac{1}{2}) = 26.57°$$



But the figure also shows that $d = (a/2)\cos\beta$. Since $\cos\beta = (1 + \tan^2\beta)^{-1/2} = (5/4)^{-1/2} = 2/\sqrt{5}$, we find that the separation between the planes is

$d = (a/2)(2/\sqrt{5}) = a/\sqrt{5}$. Using the given value $a = 4.00 \times 10^{-10}$ m, we find

$$d = \frac{4.00 \times 10^{-10}}{\sqrt{5}} = \underline{1.79 \times 10^{-10} \text{ m}} \qquad (1)$$

(b) For constructive superposition of the waves reflected from these planes, the Bragg condition must be satisfied: $j\lambda = 2d \sin\theta$, or

$$\lambda_j = (2d/j) \sin\theta \qquad (2)$$

The slowest electrons have the longest wavelengths, so we set $j = 1$ and $\theta = 90°$. This gives $\lambda_{max} = 2d$. Thus the minimum momentum is

$$p_{min} = \frac{h}{\lambda_{max}} = \frac{h}{2d} \qquad (3)$$

Using the value of d obtained in eq. (1), eq. (3) gives

$$cp_{min} = \frac{(3.00 \times 10^{8})(6.63 \times 10^{-34})}{2(1.79 \times 10^{-10})}$$

$$= 5.56 \times 10^{-16} \text{ J} = 3.47 \times 10^{3} \text{ eV}$$

Since $cp_{min} < 10^{-2} \cdot m_e c^2$, we can use nonrelativistic mechanics to find the minimum speed:

$$v_{min} = \frac{cp_{min}}{m_e c} = \frac{5.56 \times 10^{-16}}{(9.11 \times 10^{-31})(3.00 \times 10^{8})} = \underline{2.03 \times 10^{6} \text{ m/s}}$$

(c) Here the given electron speed is comparable to the speed of light, so relativistic mechanics must be employed to find the corresponding momentum. With $v = 2.00 \times 10^{8}$ m/s $= 2c/3$, we find

$$p = \frac{m_e v}{\sqrt{1 - (v/c)^2}} = \frac{(9.11 \times 10^{-31})(2.00 \times 10^{8})}{\sqrt{1 - (2/3)^2}}$$

$$= 2.44 \times 10^{-22} \text{ kg} \cdot \text{m/s}$$

The de Broglie wavelength of these electrons is

$$\lambda = \frac{h}{p} = \frac{6.63 \times 10^{-34}}{2.44 \times 10^{-22}} = 2.71 \times 10^{-12} \text{ m} \qquad (4)$$

In order to satisfy the Bragg condition for waves of this wavelength, the angle θ between the incident beam and the indicated planes must equal one of the values θ_j given by

$$\sin\theta_j = \frac{j\lambda}{2d}$$

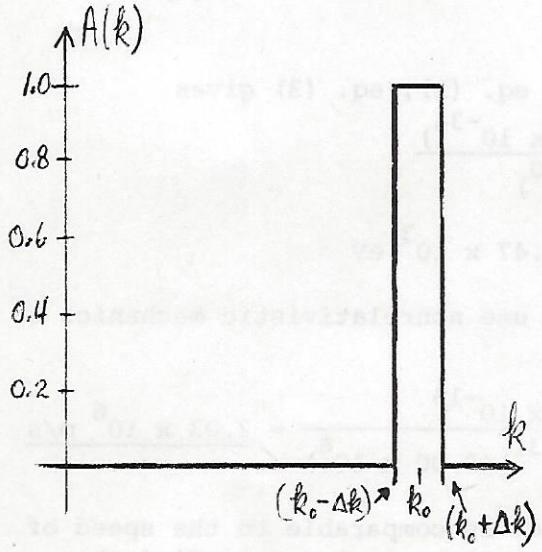
for j = 1,2,3,.... Using the values of d and λ given in eqs. (1) and (4), we find

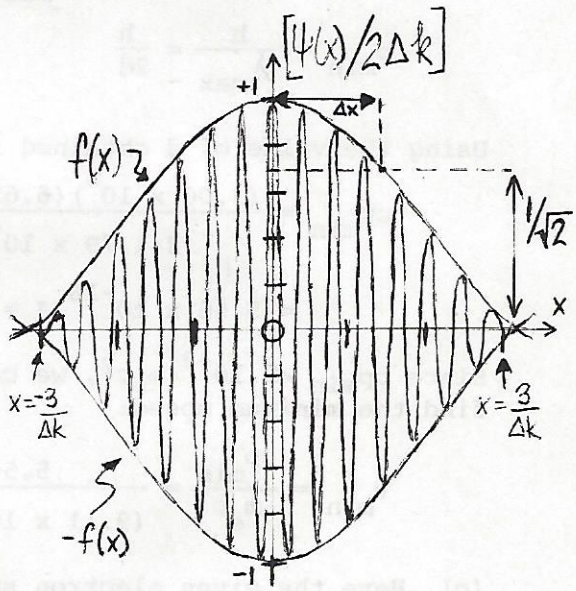
$$\theta_j = \sin^{-1}\left[\frac{j(2.71 \times 10^{-12})}{2(1.79 \times 10^{-10})}\right]$$

$$= \sin^{-1}[j(7.57 \times 10^{-3})] \qquad (5)$$

for j = 1,2,3,.... Eq. (5) has no solution for $j > (7.57 \times 10^{-3})^{-1} = 132.1$, so there are (only!) 132 possible angles. The three smallest of these are 0.434°, 0.867°, and 1.301°.

30-36





(a) The given function A(k) is shown in the graph above at the left. Using it in the expression for $\psi(x)$, we find

$$\psi(x) = \int_{k_0-\Delta k}^{k_0+\Delta k} \cos(kx)\,dk = \left[\frac{\sin(kx)}{x}\right]_{k=k_0-\Delta k}^{k=k_0+\Delta k}$$

$$= \frac{1}{x}\left\{\sin[(k_o + \Delta k)x] - \sin[(k_o - \Delta k)x]\right\} \qquad (1)$$

But

$$\sin[(k_o + \Delta k)x] = \sin(k_o x)\cdot\cos[(\Delta k)x] + \cos(k_o x)\cdot\sin[(\Delta k)x]$$

and

$$-\sin[(k_o - \Delta k)x] = -\sin(k_o x) \cdot \cos[(\Delta k)x]$$

$$+ \cos(k_o x)\ \sin[(\Delta k)x]$$

Using these in eq. (1), we obtain

$$\psi(x) = 2\,\Delta k\ \cos(k_o x) \left\{ \frac{\sin[(\Delta k)x]}{(\Delta k)x} \right\} \qquad (2)$$

The graph of this function is shown at the right on the previous page. Since $\Delta k \ll k_o$, the factor

$$f(x) \equiv \frac{\sin[(\Delta k)x]}{(\Delta k)x} \qquad (3)$$

(whose maximum value of unity occurs at $x = 0$) <u>modulates</u> the pure sinusoidal function $2\Delta k\ \cos(k_o x)$. This modulation produces the envelope shown. As in the text, we define Δx to be the distance from the maximum of $\psi(x)$ to the point at which the envelope height has been reduced by the factor $1/\sqrt{2}$. Since the maximum occurs at $x = 0$, we have

$$\frac{1}{\sqrt{2}} \equiv f(\Delta x) = \frac{\sin(\Delta k\,\Delta x)}{\Delta k\,\Delta x} \qquad (4)$$

Setting $\alpha \equiv \Delta k \Delta x$, we need to determine the value of α for which $\sin\alpha = \alpha/\sqrt{2}$. With the aid of a pocket calculator (or a table of the trigonometric functions) it is not difficult to solve this transcendental equation numerically. This result is $\alpha = 1.39$ radians, so eq. (4) implies that

$$\Delta x = \frac{1.39}{\Delta k}$$

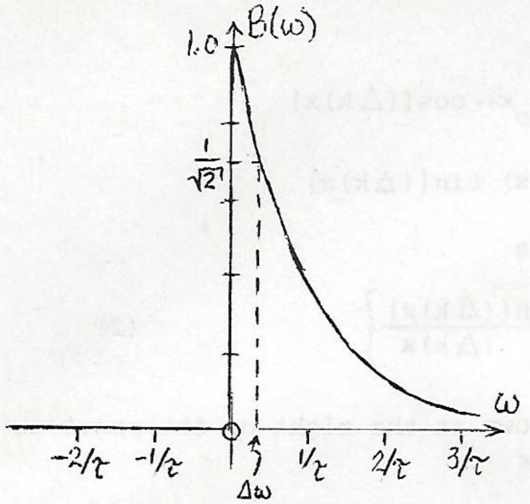
The product $\Delta x \Delta k = 1.39$ is consistent with Eq. (30-24):

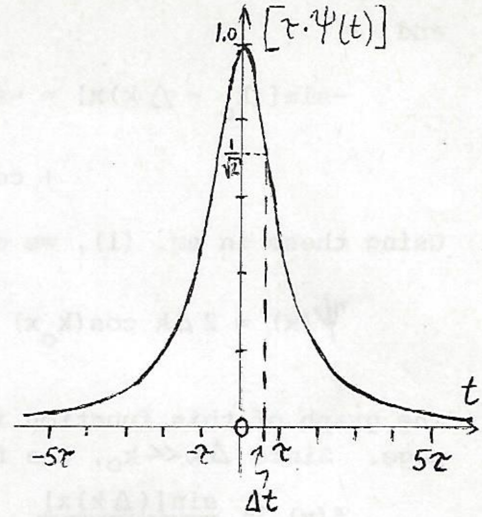
$$\Delta x \Delta k \geq \frac{1}{2}$$

(b) The given function $B(\omega)$ is shown in the graph on the next page at the left. We must evaluate

$$\psi(t) = \int_0^\infty B(\omega)\ \cos(\omega t)d\omega = \int_0^\infty e^{-\omega\tau}\ \cos(\omega t)d\omega$$

We can obtain the corresponding indefinite integral by performing two successive integrations by parts, or by consulting a table of integrals (such as the one included in the <u>CRC Handbook of Chemistry and Physics).</u> The result is





$$\int e^{-\omega\tau} \cos(\omega t)d\omega = \left(\frac{e^{-\omega\tau}}{\tau^2 + t^2}\right) [t \sin(\omega t) - \tau \cos(\omega t)]$$

Therefore the required definite integral is

$$\psi(t) = \int_0^\infty e^{-\omega\tau} \cos(\omega t)d\omega$$

$$= \left\{\left(\frac{e^{-\omega\tau}}{\tau^2 + t^2}\right) [t \sin(\omega t) - \tau\cos(\omega t)]\right\}\Bigg|_{\omega=0}^{\omega\to\infty}$$

$$= \frac{\tau}{\tau^2 + t^2} \qquad (5)$$

The graph of this function is shown above at the right. We notice that $\psi(t)$ is an even function and that its maximum value occurs at $t = 0$. Adopting the definition of Δt from the text, we put

$$\psi(\Delta t) \equiv \frac{1}{\sqrt{2}} \psi(0)$$

Using eq. (5), we have

$$\frac{\tau}{\tau^2 + (\Delta t)^2} = \frac{1}{\sqrt{2}}\left(\frac{\tau}{\tau^2}\right)$$

from which we find

$$\tau^2 + (\Delta t)^2 = \sqrt{2}\, \tau^2$$

so that

$$\Delta t = (\sqrt{2} - 1)^{1/2}\tau = 0.644\tau \qquad (6)$$

The function $B(\omega)$ is maximized for $\omega = 0$, but it is not an even function. However, if we continue to define $\Delta\omega$ by

$$B(\Delta\omega) = \frac{1}{\sqrt{2}} B(0)$$

we obtain

$$e^{-\tau\Delta\omega} = \frac{1}{\sqrt{2}}$$

so that

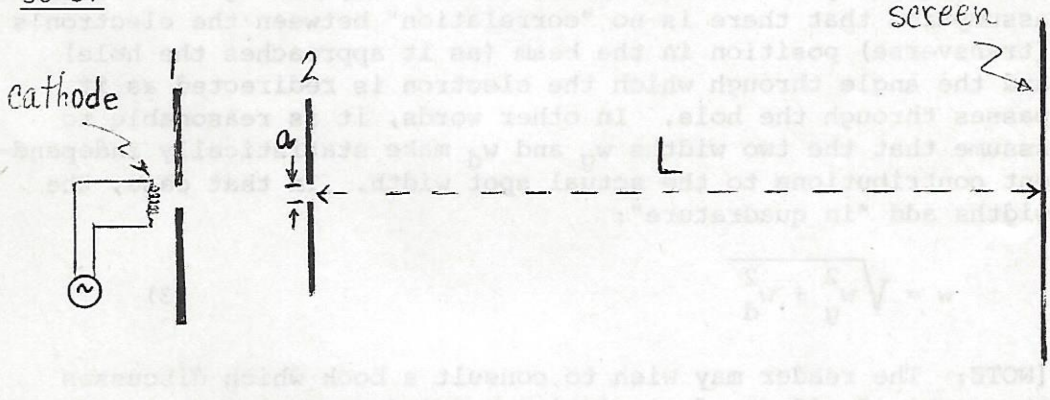
$$\Delta\omega = \frac{\ln\sqrt{2}}{\tau} = \frac{\ln 2}{2\tau} = \frac{0.347}{\tau} \qquad (7)$$

Equations (6) and (7) imply that

$$\Delta\omega\Delta t = \left(\frac{0.347}{\tau}\right)\cdot(0.644\tau) = 0.223$$

This value for the product $\Delta\omega\Delta t$ violates Eq. (30-25), although not by a large factor. The reason for this result is traceable to the definition we have used for the width of a function. If we had used the more complicated definition that is customary in advanced work, the product $\Delta\omega\Delta t$ would not have violated Eq. (30-25). With the simple definition of function width that is given in the text, we must expect occasional (mild) violations of Eq. (30-24) and/or Eq. (30-25), depending on the specific functions involved.

30-37



(a) The figure above illustrates the situation. Since the kinetic energy gained by an electron between the two metal plates is much greater than the typical kinetic energy with which an electron passes through the first hole, the exact size of the first hole is irrelevant as long as the electron beam fills the second hole. In the absence of diffraction (and ignoring any random velocities of beam electrons in the plane at right angles to the axis of the "gun"), the spot on the screen would have a width equal to the hole diameter a. Referring to this as the "geometrical" spot diameter w_g, we have

$$w_g = a \qquad (1)$$

Because the electrons have a wavelike character, there is diffraction as the beam passes through the hole. Assuming that the de Broglie wavelength λ is much less than the hole diameter, the angular half-width of the diffracted beam is given by Eq. (28-25):

$$\theta_d = \frac{1.22\lambda}{a}$$

(This is the "deflection" angle at which the first minimum occurs. As discussed in Sec. 28-9, the factor 1.22 appears because we are dealing with a round hole rather than a long narrow slit.) If diffraction were the only factor contributing to the finite width of the spot of the screen, we would have

$$w = w_d = (2\theta_d)L = \frac{2.44\lambda L}{a} \qquad (2)$$

Since both the finite hole width and the diffraction contribute to the actual spot width, we must expect that $w \gtrsim \max(w_g, w_d)$. In principle, a full analysis of the problem could be made in order to predict the actual profile of the beam at the position of the screen. The precise relationship between the spot width (defined according to some particular convention), and the quantities w_g and w_d, might prove to be quite complicated. However, we can quickly obtain an expression for w by making the reasonable assumption that there is no "correlation" between the electron's (transverse) position in the beam (as it approaches the hole) and the angle through which the electron is redirected as it passes through the hole. In other words, it is reasonable to assume that the two widths w_g and w_d make statistically independent contributions to the actual spot width. In that case, the widths add "in quadrature":

$$w = \sqrt{w_g^2 + w_d^2} \qquad (3)$$

[NOTE: The reader may wish to consult a book which discusses the combined effect of coexisting but independent sources of error in a quantity. See, for example, Statistical Treatment of Experimental Data, by Hugh D. Young (McGraw-Hill Book Co., 1962)]. Using eqs. (1) - (3), we find

$$w^2 = w_g^2 + w_d^2 = a^2 + \left(\frac{2.44\lambda L}{a}\right)^2 \tag{4}$$

The width w is minimized by the same value of a that minimizes w^2, so we set

$$0 = \frac{d(w^2)}{da} = 2a - \frac{2(2.44\lambda L)^2}{a^3} \tag{5}$$

According to eq. (5), the spot width is minimized when

$$a^4 = (2.44\lambda L)^2$$

or when

$$a \equiv a_o = \sqrt{2.44\lambda L} \tag{6}$$

We notice that when $a = a_o$, we have

$$w_g = a_o = \sqrt{2.44\lambda L}$$

and

$$w_d = \frac{2.44\lambda L}{a_o} = \sqrt{2.44\lambda L}$$

Thus, the resultant spot width is minimized when the contributing widths are equal, as stated in the hint. Using eqs. (4) and (6), we find that the minimum spot width is

$$w_{min} = a_o\sqrt{2} = \sqrt{4.88\lambda L} \tag{7}$$

(b) Reasonable values for the accelerating voltage and the free-flight distance in a television tube are V = 5000 volts and L = 30 cm = 0.3 m. Then the de Broglie wavelength of the accelerated electrons is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_e eV}}$$

$$= \frac{(6.63 \times 10^{-34})}{\sqrt{2(9.11 \times 10^{-31})(1.60 \times 10^{-19})(5 \times 10^3)}}$$

$$= 1.74 \times 10^{-11}\ m$$

Using eq. (6), we find that the optimum hole diameter is

$$a_o = \sqrt{(2.44)(1.74 \times 10^{-11})(0.3)}$$

$$= 3.56 \times 10^{-6} \simeq \underline{3.6 \times 10^{-6}\ \text{m}}$$

The corresponding spot width is given by eq. (7):

$$w_{min} = a_o\sqrt{2} \simeq \underline{5.1 \times 10^{-6}\ \text{m}}$$

30-38

(a) We suppose that the student drops the balls from positions that are uniformly distributed over a horizontal rectangular region of width Δx (and length much greater than Δx). We assume that this region is precisely centered over the crack in the sidewalk, with its long dimension parallel to the crack. As the balls are dropped, the position-momentum uncertainty principle implies that the x component of the momentum is uncertain by at least $h/\Delta x$:

$$\Delta p_x \gtrsim \frac{h}{\Delta x}$$

Since she is attempting to drop the balls vertically, the x component of momentum is distributed symmetrically about $p_x = 0$, but the above inequality implies an uncertainty in the x component of velocity:

$$\Delta v_x = \frac{\Delta p_x}{m} \gtrsim \frac{h}{m\Delta x} \qquad (1)$$

The x coordinate of the impact point will be uncertain for two reasons. The first contribution is due to the range in release positions. If this were the only source of uncertainty in the impact location, we would have a spread

$$s_g = \Delta x \qquad (2)$$

However, there is an additional spread introduced by the velocity uncertainty given in eq. (1). After the fall time t, this velocity uncertainty implies a spread

$$s_u = (\Delta v_x)t \gtrsim \frac{ht}{m\Delta x} \qquad (3)$$

(Here the subscript u refers to the position-momentum uncertainty principle.) Assuming that s_g and s_u make statistically independent contributions to the overall impact spread, we have

$$s = \sqrt{s_g^2 + s_u^2} \qquad (4)$$

(NOTE: In the solution for Exercise 30-37, a similar statement and equation are presented, with fuller justification than is given here. It is left to the reader to construct the appropriate argument for the present exercise.) The impact spread s is minimized by the same value of the "launch window" Δx that minimizes s^2, so we set

$$0 = \frac{d(s^2)}{d(\Delta x)} = \frac{d}{d(\Delta x)}(s_g^2 + s_u^2) = \frac{d}{d(\Delta x)}\left[(\Delta x)^2 + \frac{(ht/m)^2}{(\Delta x)^2}\right]$$

$$= 2\Delta x - \frac{2(ht/m)^2}{(\Delta x)^3} \qquad (5)$$

Hence the impact spread is minimized for

$$(\Delta x)^4 = (ht/m)^2$$

or when

$$\Delta x \equiv \Delta x_o = \sqrt{ht/m}$$

For this value of Δx, eqs. (2) - (4) imply that

$$s_g = s_u = \Delta x_o$$

and

$$s_{min} = \sqrt{s_g^2 + s_u^2} = \Delta x_o\sqrt{2} = \sqrt{2ht/m} \qquad (6)$$

But the free-fall time t is given by $t = \sqrt{2\ell/g}$, so eq. (6) implies that the minimum impact spread is

$$s_{min} = \sqrt{\frac{2h}{m}\sqrt{\frac{2\ell}{g}}} = 2^{3/4}\left(\frac{h}{m}\right)^{1/2}\left(\frac{\ell}{g}\right)^{1/4}$$

Hence the balls will miss the crack by approximately

$$\frac{s_{min}}{2} \simeq 2^{-1/4}\left(\frac{h}{m}\right)^{1/2}\left(\frac{\ell}{g}\right)^{1/4} \qquad (7)$$

This establishes the desired result.

(b) Using m = 0.3 kg, ℓ = 10 m, and g = 9.80 m/s^2, eq. (7) predicts that the balls will typically miss the crack by

$$2^{-1/4}\left(\frac{6.63 \times 10^{-34}}{0.3}\right)^{1/2}\left(\frac{10}{9.8}\right)^{1/4} \simeq 4 \times 10^{-17} \text{ m}$$

which is about one one-hundredth of the diameter of a large nucleus! The corresponding optimum launch window $\Delta x_0 = s_{min}/\sqrt{2}$ is of the same order of magnitude. Our result makes it clear that the uncertainty principle is not involved in setting a practical limit to the success of the student's attempts to hit the crack.

CHAPTER THIRTY-ONE

31-6

As stated in the exercise, we have

$$P(r) = Kr^2 e^{-2r/r_1}$$

where K is a constant. The range of r is $0 \leq r < \infty$. Since P(r) is nonnegative for all r and since $P(r) \to 0$ as $r \to 0$ and as $r \to \infty$, the maximum must occur at a value of r for which the slope vanishes. Hence we set

$$0 = \frac{dP}{dr} = 2Kre^{-2r/r_1} + Kr^2 \cdot \left(-\frac{2}{r_1}\right) \cdot e^{-2r/r_1}$$

$$= 2Kr\left(1 - \frac{r}{r_1}\right) \cdot e^{-2r/r_1}$$

This equation has only one solution: $r = r_1$, which must therefore be the location of the maximum.

31-7

The natural frequency ν of oscillation of a pair of identical particles (each of mass m) connected by a Hooke's law spring of force constant k is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$$

where μ is the reduced mass. (See Sec. 11-4.) In terms of the individual masses m, the reduced mass $\mu = (m)(m)/(m + m) = \frac{1}{2}m$, so that

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{(m/2)}} = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} \qquad (1)$$

The lowest possible energy associated with motion along the line to separation is given by Eq. (31-40), with n = 1:

$$E_1 = \frac{1}{2}h\nu = \frac{h}{4\pi}\sqrt{\frac{2k}{m}} \qquad (2)$$